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Inverse source problems: support reconstruction, uncertainty principles, and stability

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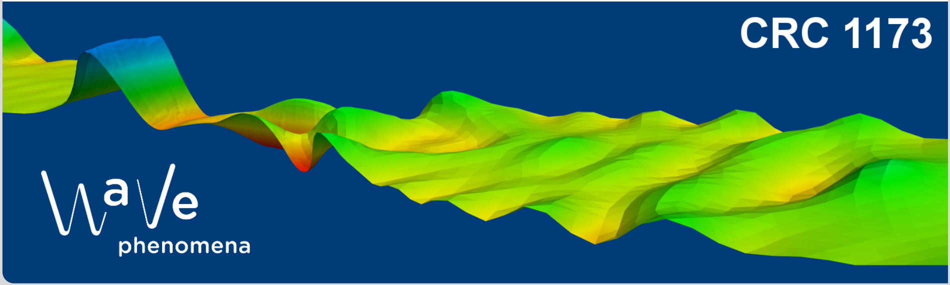
IPwin2021: Virtual PhD Winter School at DTU, January 25-29, 2021



Outline

- Direct source problem
- Inverse source problem
  - non-radiating sources
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- The source-to-far field map
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  - properties of the singular values
  - subspaces of non-evanescent far fields
- Far field splitting and data completion
  - problem statement, uniqueness, ill-posedness
  - uncertainty principles
  - stability estimates

References

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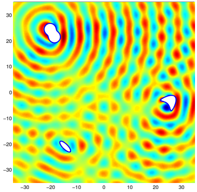
Direct source problem

Direct source problem

We consider **time-harmonic wave propagation** in $\mathbb{R}^2$.



$g \in L^2_0(\mathbb{R}^2)$ ... source term

$k > 0$ ... wave number

Source problem: Find $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$\Delta v + k^2 v = -k^2 g \quad \text{in } \mathbb{R}^2$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial v}{\partial |x|} - \mathrm{i}kv\right) = 0$$ Sommerfeld radiation condition (SRC)

Direct source problem

Interpretation in the time domain:

$$c > 0 \ \ldots \quad \text{wave speed}$$

$$\omega = \frac{ck}{2\pi} \ \ldots \quad \text{frequency}$$

$$\lambda = \frac{c}{\omega} = \frac{2\pi}{k} \ \ldots \quad \text{wave length}$$

Then

$$G(x,t) = \mathrm{Re}\left(g(x)e^{-\mathrm{i}\omega t}\right), \qquad x \in \mathbb{R}^2,\ t \in \mathbb{R}$$

$$V(x,t) = \mathrm{Re}\left(v(x)e^{-\mathrm{i}\omega t}\right), \qquad x \in \mathbb{R}^2,\ t \in \mathbb{R}$$

satisfy

$$\frac{\partial^2 V}{\partial t}(x,t) - c^2 \Delta V(x,t) = -c^2 G(x,t) \qquad \text{in } \mathbb{R}^2$$

Direct source problem

Rescale everything to $k = 1$, i.e., $\lambda = 2\pi$ is the characteristic length scale.

"measure distances in wave lenghts"

Let

$$u(x) := v(kx) \quad \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$$
$$f(x) := g(kx) \quad \in L^2_0(\mathbb{R}^2)$$

then

$$\Delta u + u = -f \qquad \text{in } \mathbb{R}^2$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u}{\partial |x|} - \mathrm{i}u\right) = 0 \qquad \text{Sommerfeld radiation condition (SRC)}$$

Direct source problem

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"measure distances in wave lenghts"

$$\Delta u + u = -f \qquad \text{in } \mathbb{R}^2$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u}{\partial |x|} - \mathrm{i}u\right) = 0 \qquad \text{Sommerfeld radiation condition (SRC)}$$

i.e.,

$$\int_{\mathbb{R}^2} (\nabla u \cdot \nabla \psi - u\psi)\,\mathrm{d}x = \int_{\mathbb{R}^2} f\psi\,\mathrm{d}x \qquad \text{for all } \psi \in H_0^1(\mathbb{R}^2)$$

Away from $\operatorname{supp}(f)$ weak solutions are classical solutions, and SRC makes sense.

Definition (Radiating solution)

Solutions to the direct source problem that satisfy SRC are radiating solutions.

Direct source problem

Bessel's equation: For $n = 0, 1, 2, \dots$

$$y''(r) + \frac{1}{r}y'(r) + \left(1 - \frac{n^2}{r^2}y(r)\right) = 0\,, \qquad r > 0\,,$$

has two linearly independent solutions J_n and Y_n (**Bessel functions**).

Negative index:

$$J_{-n} = (-1)^n J_n \qquad \text{and} \qquad Y_{-n} = (-1)^n Y_n$$

Second family of linearly independent solutions (**Hankel functions**):

$$H_n^{(1)} = J_n + \mathrm{i}\,Y_n \qquad \text{and} \qquad H_n^{(2)} = J_n - \mathrm{i}\,Y_n$$

Asymptotic expansion for large argument:

$$H_n^{(1)}(r) = \sqrt{\frac{2}{\pi r}} e^{\mathrm{i}\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right)} + O\left(r^{-\frac{3}{2}}\right) \qquad \text{as } r \to \infty$$

Direct source problem

Fundamental solution: Let

$$\Phi(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(|x-y|)\,, \qquad x,y \in \mathbb{R}^2,\ x \neq y\,,$$

then

$$\Delta_x \Phi(x,y) + \Phi(x,y) = 0\,, \qquad x,y \in \mathbb{R}^2,\ x \neq y$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial \Phi(x,y)}{\partial |x|} - \mathrm{i}\Phi(x,y)\right) = 0\,, \qquad \text{SRC}$$

Note that, as $|x-y| \to 0$,

$$\Phi(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + O(1)\,.$$

Direct source problem

Theorem (Volume potential)

Let $f \in L^2_0(\mathbb{R}^2)$, *and*

$$(Vf)(x) := \int_{\mathbb{R}^2} \Phi(x,y) f(y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^2\,.$$

Then,

(a) $w := Vf \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$

(b) *w is a radiating solution to*

$$-\Delta w - w = f \qquad \text{in } \mathbb{R}^2$$

(c) *w is real analytic in* $\mathbb{R}^2 \setminus \mathrm{supp}(f)$

Proof: See, e.g.,

A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Springer, New York, 2011

$\square$

Direct source problem

Proposition (Uniqueness)

Let $f \in L^2_0(\mathbb{R}^2)$. Then $u := Vf$ is the unique radiating solution to the direct source problem

$$\Delta u + u = -f \qquad \text{in } \mathbb{R}^2$$

Proof:

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$\square$

Corollary

(a) *Entire radiating solutions to $\Delta u + u = 0$ in $\mathbb{R}^2$ are zero.*

(b) *The unique radiating solution to $\Delta u + u = -f$ in $\mathbb{R}^2$, $f \in L^2_0(\mathbb{R}^2)$, is real analytic in $\mathbb{R}^2 \setminus \operatorname{supp}(f)$.*

(c) *If $\Delta u + u = 0$ on a connected set $\Omega \subset \mathbb{R}^2$, and $u \equiv 0$ on an open subset of Ω, then $u \equiv 0$ on all of Ω.*

Direct source problem

Theorem (Rellich's lemma)

Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{B_R(0)})$ *be a radiating solution to*

$$\Delta u + u = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{B_R(0)}$$

satisfying

$$\lim_{r\to\infty} \int_{|y|=r} |u(y)|^2 \, \mathrm{d}s(y) = 0\,.$$

Then $u \equiv 0$ *in* $\mathbb{R}^2 \setminus \overline{B_R(0)}$*.*

Proof:

Direct source problem

□

Direct source problem

Far field expansion:

As $|x| \to \infty$ we have that

$$H_0^{(1)}(|x-y|) = \sqrt{\frac{2}{\pi|x-y|}}\, e^{\mathrm{i}\left(|x-y|-\frac{\pi}{4}\right)} + O\left(|x-y|^{-\frac{3}{2}}\right)$$

and using the notation $\widehat{x} = \frac{x}{|x|}$,

$$|x-y| = |x| - \widehat{x}\cdot y + O\left(|x|^{-1}\right) ,$$

$$e^{\mathrm{i}|x-y|} = e^{\mathrm{i}|x|} e^{-\mathrm{i}\widehat{x}\cdot y} \left(1 + O\left(|x|^{-1}\right)\right) ,$$

$$\sqrt{\frac{1}{|x-y|}} = \frac{1}{\sqrt{|x|}} \left(1 + O\left(|x|^{-\frac{3}{2}}\right)\right) .$$

Direct source problem

Thus, the radiating solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ of

$$\Delta u + u = -f \qquad \text{in } \mathbb{R}^2$$

with $f \in L^2_0(\mathbb{R}^2)$ satisfies

$$u(x) = \int_{\mathbb{R}^2} \Phi(x,y) f(y)\, \mathrm{d}y = \frac{e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi}} \frac{e^{\mathrm{i}|x|}}{\sqrt{|x|}} \int_{\mathbb{R}^2} e^{-\mathrm{i}\hat{x}\cdot y} f(y)\, \mathrm{d}y + O\left(|x|^{-\frac{3}{2}}\right)$$

Definition (Radiated far field)

The function $u^\infty \in L^2(S^1)$ given by

$$u^\infty(\hat{x}) := \widehat{f}(\hat{x}) := \int_{\mathbb{R}^2} e^{-\mathrm{i}\hat{x}\cdot y} f(y)\, \mathrm{d}y\,, \qquad \hat{x} \in S^1\,,$$

is the far field radiated by f.

Remark

The far field u^∞ is real analytic on S^1.

Direct source problem

Corollary (of Rellich's lemma)

If $u^\infty = 0$, *then* $u = 0$ *in* $\mathbb{R}^2 \setminus \operatorname{supp}(f)$.

Proof:

$\square$

Inverse source problem

Inverse source problem

Suppose that $f \in L^2_0(\mathbb{R}^2)$, and let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ be the radiating solution of

$$\Delta u + u = -f \qquad \text{in } \mathbb{R}^2$$

with far field $u^\infty = \widehat{f}|_{S^1}$.

Task: Deduce useful information about f from observations of u^∞.

Definition (Radiated wave/far field)

We call

(a) $u|_{\mathbb{R}^2 \setminus \mathrm{supp}(f)}$ the wave radiated by f

(b) u^∞ the far field radiated by f

Definition (Non-radiating source)

A source $f \in L^2_0(\mathbb{R}^2)$ is non-radiating, if the far field radiated by f vanishes or equivalently if the radiated wave vanishes in $\mathbb{R}^2 \setminus \mathrm{supp}(f)$.

Inverse source problem

Example (Non-radiating source)

Let $u \in C_0^\infty(\mathbb{R}^2)$, and define

$$f := -\Delta u - u \in C_0^\infty(\mathbb{R}^2)\,.$$

Then f is non-radiating (because u is the radiating solution to $\Delta u + u = -f$ in $\mathbb{R}^2$).

$\Diamond$

Definition (Equivalent sources)

Two sources $f_1, f_2 \in L_0^2(\mathbb{R}^2)$ are equivalent if they radiate the same far field.

Definition (D-free waves)

Let $D \subset \mathbb{R}^2$. Then $v \in H^1(D)$ is D-free if $\Delta v + v = 0$ in D.

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Theorem

Let $D \subset \mathbb{R}^2$ open and bounded with sufficiently smooth boundary. A source $f \in L^2(D)$ is non-radiating if and only if either of the conditions below holds:

(a) $f = -\Delta w - w$ *for some* $w \in H_0^2(D)$,

(b) f *is* L^2*-orthogonal to all* D*-free waves.*

Here, $H_0^2(D) = \left\{ u|_D \;\middle|\; u \in H^2(\mathbb{R}^2),\ u \equiv 0 \text{ in } \mathbb{R}^2 \setminus \overline{D} \right\}$

Proof:

Inverse source problem

Inverse source problem

Inverse source problem

Remark (Upper bounds on source support of f)

If $f \in L^2_0(\mathbb{R}^2)$ radiates u^∞, and $\operatorname{supp}(f) \subset D \subset\subset B_R(0)$, then we can choose $\phi_\varepsilon \in C^\infty(\mathbb{R}^2)$ such that, for $\varepsilon > 0$,

$$\phi_\varepsilon(x) = \begin{cases} 0\,, & x \in B_R(0) \\ 1\,, & x \in \mathbb{R}^2 \setminus B_{R+\varepsilon(0)} \end{cases}$$

Let u be the radiating solution to $\Delta u + u = -f$ in $\mathbb{R}^2$, and define

$$f_\varepsilon := -\Delta(\phi_\varepsilon u) - (\phi_\varepsilon u) \in C_0^\infty(\mathbb{R}^2)\,.$$

Then f_ε radiates u^∞ but

$$\operatorname{supp}(f_\varepsilon) \cap \operatorname{supp}(f) = \emptyset\,.$$

*In particular we **cannot deduce upper bounds** for $\operatorname{supp}(f)$ from u^∞.*

Inverse source problem

Definition (Far field carrier)

A compact set $M \subset \mathbb{R}^2$ **carries** a far field u^∞, if any open neighborhood of M supports a source $f \in L^2_0(\mathbb{R}^2)$ that radiates that far field.

Example

Inverse source problem

Inverse source problem

Lemma (Far field carrier)

Suppose $M \subset \mathbb{R}^2$ is compact, and $\mathbb{R}^2 \setminus M$ is connected. Then the following conditions are equivalent:

(a) *M carries a far field u^∞.*

(b) *There exists a unique radiating solution to $\Delta u + u = 0$ in $\mathbb{R}^2 \setminus M$, which has far field u^∞.*

Proof:

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Inverse source problem

□

Inverse source problem

Lemma (Intersecting far field carriers)

Let $M_1, M_2 \subset \mathbb{R}^2$ *be compact and carry a far field* u^∞. *Suppose that*

$$\mathbb{R}^2 \setminus M_1, \quad \mathbb{R}^2 \setminus M_2 \quad \text{and} \quad \mathbb{R}^2 \setminus (M_1 \cup M_2) \quad \text{are connected.}$$

Then $M_1 \cap M_2$ *carries* u^∞.

Proof:

Inverse source problem

□

Inverse source problem

Definition (Far field support class)

A **far field support class** is a collection $\mathcal{M}$ of compact sets in $\mathbb{R}^2$ that satisfies

(a) $\mathcal{M}$ is closed under intersection

(b) $\forall M \in \mathcal{M}$: $\mathbb{R}^2 \setminus M$ is connected

(c) $\forall M_1, M_2 \in \mathcal{M}$: $\mathbb{R}^2 \setminus (M_1 \cup M_2)$ is connected

Definition ($\mathcal{M}$-support)

Let $\mathcal{M}$ be a far field support class. The **$\mathcal{M}$-support** of a far field u^∞ is

$$M_{u^\infty} = \bigcap_{\substack{M \in \mathcal{M} \\ M \text{ carries } u^\infty}} M\,.$$

Theorem

M_{u^∞} is the smallest set in $\mathcal{M}$ that carries u^∞.

Inverse source problem

Proof:

Inverse source problem

□

Inverse source problem

Proposition and Definition (Convex source support)

(a) *The class of* ***compact convex subsets*** *of* $\mathbb{R}^2$ *is a far field support class.*

(b) *The corresponding* $\mathcal{M}$*-support of a far field* u^∞ *is called the* ***convex source support*** $\operatorname{c\,supp}(u^\infty)$ *of* u^∞*.*

Corollary

The convex source support $\operatorname{c\,supp}(u^\infty)$ *is the* ***smallest convex set*** *that carries* u^∞*.*

Example

Inverse source problem

The source-to-far field map

The source-to-far field map

Restricted Fourier transform:

$$\mathcal{F} : L^2_0(\mathbb{R}^2) \to L^2(S^1)\,, \qquad \mathcal{F}f := \widehat{f}|_{S^1}$$

We have seen that the null space of $\mathcal{F}$ is

$$\mathcal{N}(\mathcal{F}) = \left\{ g = -\Delta v - v \;\middle|\; v \in H^2_0(\mathbb{R}^2) \right\} .$$

These are non-radiating sources.

Sources supported in $B_R(0)$:

$$\mathcal{F}_{B_R(0)} : L^2\left(B_R(0)\right) \to L^2(S^1)\,, \qquad \mathcal{F}_{B_R(0)}f := \widehat{f}|_{S^1}$$

Since $\mathcal{F}_{B_R(0)}$ is an integral operator with smooth kernel, it is compact.

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Theorem (SVD of $\mathcal{F}_{B_R(0)}$)

The singular value decomposition of $\mathcal{F}_{B_R(0)}$ *is given by*

$$\mathcal{F}_{B_R(0)} f = \sum_{n=-\infty}^{\infty} \sigma_n(R) \langle f, v_n \rangle_{L^2(B_R(0))} u_n \,,$$

where

$$\sigma_n^2(R) = (2\pi)^2 \int_0^R J_n^2(r) r \, \mathrm{d}r$$

$$u_n(\widehat{x}) = \frac{e^{\mathrm{i}n\varphi_x}}{\sqrt{2\pi}} \,, \qquad \widehat{x} = (\cos\varphi_x, \sin\varphi_x) \in S^1$$

$$v_n(y) = \frac{\sqrt{2\pi}\mathrm{i}^n e^{\mathrm{i}n\varphi_y} J_n(|y|)}{\sigma_n} \,, \qquad y = |y|(\cos\varphi_y, \sin\varphi_y) \in B_R(0)$$

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Proof:

The source-to-far field map

The source-to-far field map

□

The source-to-far field map

Remark

(a) *The left singular values $\{u_n\}_n$ are the Fourier basis of $L^2(S^1)$.*

(b) *Given $\alpha \in L^2(S^1)$, we can expand*

$$\alpha(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{e^{\mathrm{i}n\vartheta}}{\sqrt{2\pi}}\,, \qquad \theta = (\cos\vartheta, \sin\vartheta) \in S^1\,,$$

where

$$\alpha_n = \int_{S^1} \alpha(\theta) \frac{e^{-\mathrm{i}n\vartheta}}{\sqrt{2\pi}}\, \mathrm{d}s(\theta)\,, \qquad n \in \mathbb{Z}\,.$$

(c) *We will often identify $\alpha \in L^2(S^1)$ with its Fourier coefficients $\{\alpha_n\}_n \in \ell^2$. Parseval's identity reads*

$$\|\alpha\|_{L^2(S^1)} = \|\{\alpha_n\}_n\|_{\ell^2}\,.$$

The source-to-far field map

The following theorem connects the size of the support of a source f with the decay behavior of the Fourier coefficients $\{u_n^\infty\}_n$ of its radiated far field u^∞.

Theorem

Let $\alpha \in L^2(S^1)$*. Then*

$$\alpha \in \mathcal{R}(\mathcal{F}_{B_R(0)}) \qquad \Longleftrightarrow \qquad \sum_{n=-\infty}^{\infty} \frac{|\alpha_n|^2}{\sigma_n^2(R)} < \infty\,.$$

Proof:

$\square$

The source-to-far field map

If $\alpha \in \mathcal{R}(\mathcal{F}_{B_R(0)})$, then the source

$$f_\alpha^*(x) := \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{\sigma_n(R)} v_n(x)\,, \qquad x \in B_R(0)\,,$$

is well-defined, and $f_\alpha^* \in L^2(B_R(0))$ is the source with smallest L^2-norm

$$\|f_\alpha^*\|_{L^2(B_R(0))} = \left(\sum_{n=-\infty}^{\infty} \frac{|\alpha_n|^2}{\sigma_n^2(R)} \right)^{\frac{1}{2}}$$

that is supported in $B_R(0)$ and radiates α.

Definition (Minimal power source)

Let $\alpha \in \mathcal{R}(\mathcal{F}_{B_R(0)})$. We call f_α^* the **minimal power source** supported in $B_R(0)$ that radiates α.

The source-to-far field map

If $f \in L^2(B_R(z))$, $z \in \mathbb{R}^2$, then the far field radiated by f is

$$\begin{aligned} u^\infty(\widehat{x}) &= \int_{B_R(z)} f(y) e^{-\mathrm{i}\widehat{x}\cdot y} \,\mathrm{d}y \\ &= \int_{B_R(0)} f(y+z) e^{-\mathrm{i}\widehat{x}\cdot(y+z)} \,\mathrm{d}y \\ &= e^{-\mathrm{i}z\cdot\widehat{x}} \mathcal{F}_{B_R(0)} f(\cdot + z)\,, \end{aligned}$$

and $f(\cdot + z) \in L^2(B_R(0))$.

Definition (Far field translation operator)

Let $z \in \mathbb{R}^2$. Then $T_z : L^2(S^1) \to L^2(S^1)$,

$$(T_z\alpha)(\theta) := e^{-\mathrm{i}z\cdot\theta}\alpha(\theta)\,,$$

is called the **far field translation operator**.

The source-to-far field map

Far field translation operator:

$$T_z : L^2(S^1) \to L^2(S^1)\,, \qquad (T_z\alpha)(\theta) \;=\; e^{-\mathrm{i}z\cdot\theta}\alpha(\theta)$$

Using the Jacobi-Anger-expansion

$$e^{\pm\mathrm{i}\theta\cdot y} \;=\; \sum_{n=-\infty}^{\infty} (\pm\mathrm{i})^n e^{-\mathrm{i}n\varphi_y} J_n(|y|) e^{\mathrm{i}n\vartheta}$$

we can identify T_z with a **convolution operator** acting on the Fourier coefficients

$$T_z : \ell^2 \to \ell^2\,, \qquad T_z\{\alpha_n\}_n \;=\; \left\{ \sum_{n=-\infty}^{\infty} \alpha_{m-n}(-\mathrm{i})^n J_n(|z|) e^{-\mathrm{i}n\varphi_z} \right\}_m$$

Let $\{\alpha_m^z\}_m$ denote the Fourier coefficients of $T_z\alpha$.

The source-to-far field map

Corollary

Let $\alpha \in L^2(S^1)$ *and* $z \in \mathbb{R}^2$*. Then*

$$\alpha \in \mathcal{R}(\mathcal{F}_{B_R(z)}) \quad \Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} \frac{|\alpha_n^{-z}|^2}{\sigma_n^2(R)} < \infty .$$

Proof:

The source-to-far field map

$\square$

The source-to-far field map

Algorithm to approximate $\operatorname{c\,supp}(\alpha)$:

Given: $u^\infty \in L^2(S^1)$

- Choose origins $z_1, \dots, z_L$

For $\ell = 1, \dots, L$

- Compute Fourier coefficients $\{\alpha_n^{-z_\ell}\}_n \in \ell^2$ of $T_{-z_\ell}\alpha$
- Find smallest radius $R_\ell > 0$ such that

$$\sum_{n=-\infty}^{\infty} \frac{|\alpha_n^{-z_\ell}|^2}{\sigma_n^2(R_\ell)} < \infty$$

$\rightsquigarrow$ α is carried by $B_{R_\ell}(z_\ell)$!

End For

- $\operatorname{c\,supp}(u^\infty) \approx \bigcap_{\ell=1}^{L} \overline{B_{R_\ell}(z_\ell)}$

The source-to-far field map

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Properties of the singular values $\sigma_n(R)$ of $\mathcal{F}_{B_R(0)}$:

$$\sigma_n^2(R) = 2\pi \left(2\pi \int_0^R J_n^2(r) r \,\mathrm{d}r \right)$$

Explicit formula:

$$s_n^2(R) = R^2 \pi \left(J_n^2(R) - J_{n-1}(R) J_{n+1}(R) \right) = \pi \left((R J_n'(R) - (R^2 - n^2) J_n^2(R) \right)$$

Lemma

$$\sum_{n=-\infty}^{\infty} s_n^2(R) = \pi R^2$$

Proof:

$\square$

The source-to-far field map

Lemma (Monotonicity)

$$s_{n-1}^2(R) - s_{n+1}^2(R) \;\geq\; 0\,, \qquad n \geq 0\,,$$

i.e., odd and even singular values are decreasing ($n \geq 0$) or increasing ($n \leq 0$).

Proof:

$\square$

The source-to-far field map

Lemma (Exponential decay for $|n| \gtrsim R$)

$$s_n^2(R) \le C \left(\frac{R^2}{n^2} e^{1-\frac{R^2}{n^2}} \right)^n \frac{R^2}{n^2} \qquad \textit{if } |n| \ge R$$

Proof: Use sharp estimates for $J_n(r)$ from

- I. Krasikov, Approximations for the Bessel and Airy functions with an explicit error term, LMS J. Comput. Math., 17:209–225, 2006

$\square$

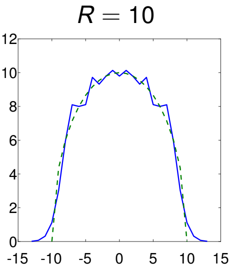
The source-to-far field map

Proposition (Asymptotic behaviour for $|n| \lesssim R$)

$$\lim_{R\to\infty} \frac{s^2_{\lceil \nu R \rceil}(R)}{2R} = \begin{cases} \sqrt{1-\nu^2} & \nu \le 1 \\ 0 & \nu \ge 1 \end{cases}$$

This means that the singular values $\sigma_n^2(R)$ of $\mathcal{F}_{B_R(0)}$ satisfy

$$\frac{\sigma_n^2(R)}{4\pi} \sim \begin{cases} \sqrt{R^2-n^2} & n \lesssim R \\ 0 & n \gtrsim R \end{cases}$$



The source-to-far field map

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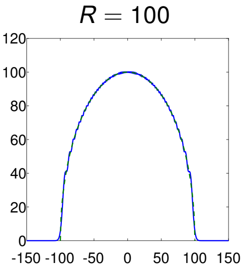

The source-to-far field map

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Theorem

Suppose that $R > n \geq 0$, *define* $\alpha \in (0, \frac{\pi}{2})$ *by* $\cos\alpha = \frac{n}{R}$, *and assume* $\alpha > \varepsilon > 0$. *Then*

$$\left| J_n(R) - \sqrt{\frac{2}{\pi R \sin\alpha}} \cos\left(R(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}\right)\right| \leq \frac{C(\varepsilon)}{R},$$

$$\left| J_n'(R) + \sqrt{\frac{2}{\pi R \sin\alpha}} \sin\alpha \sin\left(R(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}\right)\right| \leq \frac{C(\varepsilon)}{R}.$$

Idea of proof:

$\square$

The source-to-far field map

Proof (of the proposition):

$\square$

The source-to-far field map

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Regularization:

- physical sources have **limited power**

$$P > 0 \ldots \quad \text{power threshold of reasonable sources}$$

- receivers that are used to measure far fields have **limited sensitivity**

$$p > 0 \ldots \quad \text{power threshold of reasonable receivers}$$

If $\alpha \in L^2(S^1)$ is radiated from $B_R(0)$, then

$$f_\alpha^* \in L^2(B_R(0)) \text{ is the mininal power source supported in } B_R(0).$$

Suppose that

$$\|f_\alpha^*\|_{L^2(B_R(0))} \leq P\,,$$

and let

$$N = N(R, P, p) := \sup_{\sigma_n^2(R) \geq \frac{p}{P}} n\,.$$

The source-to-far field map

Then

$$P \geq \sum_{|n|>N} \frac{|\alpha_n|^2}{\sigma_n^2(R)} \geq \frac{1}{\sigma_{N+1}^2(R)} \sum_{|n|>N} |\alpha_n|^2 > \frac{P}{p} \sum_{|n|>N} |\alpha_n|^2$$

and thus

$$\sum_{|n|>N} |\alpha_n|^2 < p$$

is below the power threshold of any reasonable receiver!

Subspace of **detectable** far fields radiated by **limited power sources** in $B_R(0)$:

$$V_N^0 := \left\{ \alpha \in L^2(S^1) \mid \alpha(\widehat{x}) = \sum_{n=-N}^{N} \alpha_n e^{in\varphi_x}\,,\ \widehat{x} \in S^1 \right\}.$$

We call V_N^0 the **subspace of non-evanescent far fields** radiated from $B_R(0)$.

The source-to-far field map

Remark

Due to the exponential decay of $\sigma_n(R)$ for $n \geq R$, we have that

$$N \gtrsim R$$

is a good choice for a large range of P and p.

Far field splitting and data completion

Far field splitting and data completion

Far field splitting:

Suppose that $u^\infty = u_1^\infty + u_2^\infty$, where

$$u_1^\infty \text{ is radiated from } B_{R_1}(z_1)$$
$$u_2^\infty \text{ is radiated from } B_{R_2}(z_2)$$

with $z_1, z_2 \in \mathbb{R}^2$ and $R_1, R_2 > 0$.

Goal: Given u^∞, z_1, z_2, R_1, R_2 recover u_1^∞ and u_2^∞.

Lemma

If $B_{R_1}(z_1) \cap B_{R_2}(z_2) = \emptyset$, then

$$\mathcal{R}\left(\mathcal{F}_{B_{R_1}(z_1)}\right) \cap \mathcal{R}\left(\mathcal{F}_{B_{R_2}(z_2)}\right) = \{0\}.$$

Proof:

Far field splitting and data completion

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□

This means that

$$u_1^\infty \text{ and } u_2^\infty \text{ are uniquely determined by } u^\infty \text{ if } B_{R_1}(z_1) \cap B_{R_2}(z_2) = \emptyset.$$

On the other hand,

$$\text{both } \mathcal{R}(\mathcal{F}_{B_{R_1}(z_1)}) \text{ and } \mathcal{R}(\mathcal{F}_{B_{R_2}(z_2)}) \text{ are dense in } L^2(S^1),$$

i.e., the inverse problem to recover u_1^∞ and u_2^∞ from u^∞ is ill-posed.

We will

reconstruct the non-evanescent parts of u_1^∞ and u_2^∞,

and we show that this is **stable**.

Far field splitting and data completion

Definition (Projection onto V_N^z)

For $N \in \mathbb{N}$ let $P_N^0 : L^2(S^1) \to L^2(S^1)$ denote the orthogonal projection onto V_N^0. Then,

$$P_N^z = T_z P_N^0 T_z^*$$

is the orthonormal projection onto V_N^z, $z \in \mathbb{R}^2$.

Let $N_1 \gtrsim R_1$ and $N_2 \gtrsim R_2$.

We consider the least squares problem to find $v_1^\infty \in V_{N_1}^{z_1}$ and $v_2^\infty \in V_{N_2}^{z_2}$ such that

$$\begin{aligned} v_1^\infty + P_{N_1}^{z_1} v_2^\infty &= P_{N_1}^{z_1} u^\infty \\ P_{N_2}^{z_2} v_1^\infty + \quad v_2^\infty &= P_{N_2}^{z_2} u^\infty \end{aligned}$$

Far field splitting and data completion

This is equivalent to

$$\left(I - P_{N_1}^{z_1} P_{N_2}^{z_2}\right) v_1^\infty = P_{N_1}^{z_1} \left(I - P_{N_2}^{z_2}\right) u^\infty$$

$$\left(I - P_{N_2}^{z_2} P_{N_1}^{z_1}\right) v_2^\infty = P_{N_2}^{z_2} \left(I - P_{N_1}^{z_1}\right) u^\infty$$

and these equations are uniquely solvable if $z_1 \neq z_1$.

Below we will see that the **condition number** of these linear systems is

$$\operatorname{cond}_2 \left(I - P_{N_1}^{z_1} P_{N_2}^{z_2}\right) = \frac{1}{\sin\left(\Theta_{V_{N_1}^{z_1}, V_{N_2}^{z_2}}\right)} = \operatorname{cond}_2 \left(I - P_{N_2}^{z_2} P_{N_1}^{z_1}\right)$$

where $\Theta_{V_{N_1}^{z_1}, V_{N_2}^{z_2}}$ is the angle between $V_{N_1}^{z_1}$ and $V_{N_2}^{z_2}$.

Far field splitting and data completion

Data completion:

Suppose that u^∞ is radiated from $B_R(z)$ but cannot be observed on $\Gamma \subset S^1$. Write

$$u^\infty|_{S^1\setminus\Gamma} = u^\infty - u^\infty|_\Gamma \,.$$

Goal: Given $u^\infty|_{S^1\setminus\Gamma}$, z, R, Γ recover $u^\infty|_\Gamma$.

Since $u^\infty \in L^2(S^1)$ is real-analytic, the problem has a unique solution, if $S^1 \setminus \Gamma$ has an interior point. However, without further assumptions unique continuation is known to be an ill-posed inverse problem.

Far field splitting and data completion

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Definition (Projection onto $L^2(\Gamma)$)

For $\Gamma \subset S^1$ let $P_\Gamma : L^2(S^1) \to L^2(S^1)$

$$P_\Gamma \alpha := \alpha|_\Gamma$$

denote the orthogonal projection onto $L^2(\Gamma)$.

Let $N \gtrsim R$ and $\Gamma \subset S^1$.

We consider the least squares problem to find $v^\infty \in V_N^z$ such that

$$v^\infty + P_N^z(v^\infty|_\Gamma) = P_N^z\left(u^\infty|_{S^1\setminus\Gamma}\right)$$

$$P_\Gamma v^\infty + \quad v^\infty|_\Gamma = P_\Gamma\left(u^\infty|_{S^1\setminus\Gamma}\right)$$

Far field splitting and data completion

This is equivalent to

$$(I - P_\Gamma P_N^z)\,(v^\infty|_\Gamma) \;=\; P_N^z\left(u^\infty|_{S^1\setminus\Gamma}\right)$$

and this equation is uniquely solvable.

Below we will see that the **condition number** of this linear systems is

$$\mathrm{cond}_2\,(I - P_\Gamma P_N^z) \;=\; \frac{1}{\sin\left(\Theta_{L^2(\Gamma),V_N^z}\right)}$$

where $\Theta_{L^2(\Gamma),V_N^z}$ is the angle between $L^2(\Gamma)$ and V_N^z.

Far field splitting and data completion

The **far field translation operator**

$$T_z : L^2(S^1) \to L^2(S^1)\,, \quad (T_z\alpha)(\theta) := e^{-\mathrm{i}z\cdot\theta}\alpha(\theta)$$

acts on the Fourier coefficients $\{\alpha_n\}$ of α as a convolution operator

$$T_z : \ell^2 \to \ell^2\,, \quad (T_z\{\alpha_n\})_m = \sum_n \alpha_{m-n}\big((-\mathrm{i})^n J_n(|z|)e^{-\mathrm{i}n\varphi_z}\big)$$

We have **estimates**

$$\|T_z\alpha\|_{L^\infty(S^1)} = \|\alpha\|_{L^\infty(S^1)}$$

and

$$\begin{aligned}\|\{(T_z\alpha)_m\}_m\|_{\ell^\infty} &\leq \sup_{m\in\mathbb{Z}} \sum_{n=-\infty}^{\infty} |\alpha_{m-n}||J_n(|z|)| \leq \sup_{n\in\mathbb{Z}} |J_n(|z|)|\,\|\{\alpha_m\}_m\|_{\ell^1}\\ &\leq \frac{1}{|z|^{1/3}}\,\|\{\alpha_m\}_m\|_{\ell^1}\,.\end{aligned}$$

Far field splitting and data completion

$$\|T_c\|_{L^\infty,L^\infty} = 1 \quad \text{and} \quad \|T_c\|_{\ell^1,\ell^\infty} \leq \frac{1}{|c|^{1/3}}$$

Theorem (Uncertainty principle for far field translation)

Suppose that $\alpha_1 \in V_{N_1}^{z_1}$ *and* $\alpha_2 \in V_{N_2}^{z_2}$. *Then*

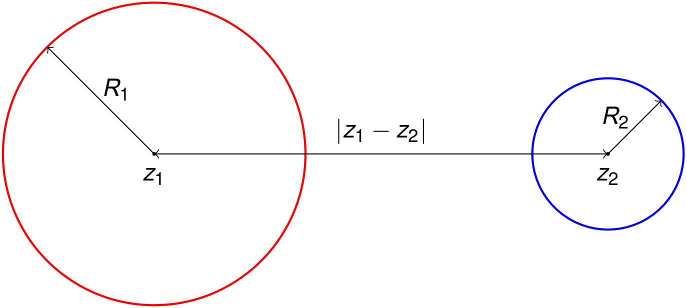
$$\frac{|\langle \alpha_1, \alpha_2 \rangle_{L^2(S^1)}|}{\|\alpha_1\|_{L^2(S^1)} \|\alpha_2\|_{L^2(S^1)}} \leq \frac{\sqrt{(2N_1+1)(2N_2+1)}}{|z_1 - z_2|^{\frac{1}{3}}} .$$

Proof:

Far field splitting and data completion

Far field splitting and data completion

$$\cos\left(\Theta_{V_{R_1}^{z_1},V_{R_2}^{z_2}}\right) := \sup_{\alpha_1\in V_{R_1}^{z_1},\alpha_2\in V_{R_2}^{z_2}} \frac{\left|\langle\alpha_1,\alpha_2\rangle_{L^2(S^1)}\right|}{\|\alpha_1\|_{L^2(S^1)}\|\alpha_2\|_{L^2(S^1)}} \le \frac{\sqrt{(2R_1+1)(2R_2+1)}}{|z_1-z_2|^{1/3}}$$



The **condition number** of the splitting operator is $\csc(\Theta_{V_{R_1}^{z_1},V_{R_2}^{z_2}})$ (!)

Far field splitting and data completion

Remark

Assuming that $N_1, N_2 \geq 1$ and

$$|z_1 - z_2| > 2(N_1 + N_2 + 1)$$

it can be shown that

$$\left\| \left\{ (T_{z_1 - z_2} \widetilde{\alpha}_1)_m \right\}_m \right\|_{\ell^\infty[-N_2, N_2]} \leq \frac{1}{|z_1 - z_2|^{\frac{1}{2}}} \left\| \left\{ (\widetilde{\alpha}_1)_n \right\}_n \right\|_{\ell^1[-N_1, N_1]}$$

and thus

$$\frac{|\langle \alpha_1, \alpha_2 \rangle_{L^2(S^1)}|}{\|\alpha_1\|_{L^2(S^1)} \|\alpha_2\|_{L^2(S^1)}} \leq \sqrt{\frac{(2N_1 + 1)(2N_2 + 1)}{|z_1 - z_2|}} .$$

This estimate is optimal.

Far field splitting and data completion

$$\|T_c\|_{L^\infty,L^\infty} = 1 \quad \text{and} \quad \|T_c\|_{\ell^1,\ell^\infty} \leq \frac{1}{|c|^{1/3}}$$

Theorem

Suppose that $\alpha \in V_N^z$ and $\beta \in L^2(\Gamma)$. Then

$$\frac{|\langle \alpha, \beta \rangle_{L^2(S^1)}|}{\|\alpha\|_{L^2(S^1)} \|\beta\|_{L^2(S^1)}} \leq \sqrt{\frac{(2N+1)|\Gamma|}{2\pi}}$$

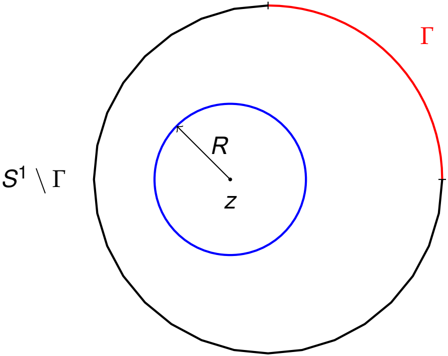
Proof:

Far field splitting and data completion

□

Far field splitting and data completion

$$\cos\left(\Theta_{V_R^z, L^2(\Gamma)}\right) := \sup_{\alpha \in V_R^z, \beta \in L^2(\Gamma)} \frac{|\langle \alpha, \beta \rangle_{L^2(S^1)}|}{\|\alpha\|_{L^2(S^1)} \|\beta\|_{L^2(S^1)}} \leq \sqrt{\frac{(2R+1)|\Gamma|}{2\pi}}$$



The **condition number** of the data completion operator is $\csc\left(\Theta_{V_R^z, L^2(\Gamma)}\right)$ (!)

Far field splitting and data completion

Lemma (Stability)

Let $V_1, V_2 \subset V$ be subspaces of a Hilbert space V. Suppose that $\alpha \in V_1$, $\beta \in V_2$ with $\cos \Theta_{V_1,V_2} \leq C < 1$, and let

$$\gamma = \alpha + \beta$$

Then

$$\|\alpha\|_V \leq (1 - C^2)^{-\frac{1}{2}} \|\gamma\|_V$$

Proof:

Far field splitting and data completion

Far field splitting and data completion

Theorem (Stability of far field splitting)

Suppose that $\gamma, \gamma^\delta \in L^2(S^1)$, $z_1, z_2 \in \mathbb{R}^2$ *and* $N_1, N_2 > 0$ *such that*

$$\frac{(2N_1+1)(2N_2+1)}{|z_1 - z_2|^{\frac{2}{3}}} < 1$$

and let

$$\gamma \overset{\text{LS}}{=} \alpha_1 + \alpha_2\,, \qquad \alpha_1 \in V_{N_1}^{z_1} \ \text{and}\ \alpha_2 \in V_{N_2}^{z_2}$$

$$\gamma^\delta \overset{\text{LS}}{=} \alpha_1^\delta + \alpha_2^\delta\,, \qquad \alpha_1^\delta \in V_{N_1}^{z_1} \ \text{and}\ \alpha_2^\delta \in V_{N_2}^{z_2}$$

Then, for $j = 1, 2$,

$$\|\alpha_j^\delta - \alpha_j\|_{L^2(S^1)}^2 \le \left(1 - \frac{(2N_1+1)(2N_2+1)}{|z_1 - z_2|^{\frac{2}{3}}}\right)^{-1} \|\gamma^\delta - \gamma\|_{L^2(S^1)}^2$$

Far field splitting and data completion

Proof:

Far field splitting and data completion

Far field splitting and data completion

Theorem (Stability of data completion)

Suppose that $\gamma, \gamma^\delta \in L^2(S^1)$, $z \in \mathbb{R}^2$, $N > 0$ *and* $\Gamma \subset S^1$ *such that*

$$\frac{(2N+1)|\Gamma|}{2\pi} < 1$$

and let

$$\gamma \overset{\mathrm{LS}}{=} \alpha + \beta\,, \qquad \alpha \in V_N^z \ \text{ and } \ \beta \in L^2(\Gamma)$$

$$\gamma^\delta \overset{\mathrm{LS}}{=} \alpha^\delta + \beta^\delta\,, \qquad \alpha^\delta \in V_N^z \ \text{ and } \ \beta^\delta \in L^2(\Gamma)$$

Then

$$\|\alpha^\delta - \alpha\|^2_{L^2(S^1)} \le \left(1 - \frac{(2N+1)|\Gamma|}{2\pi}\right)^{-1} \|\gamma^\delta - \gamma\|^2_{L^2(S^1)}$$

Proof: Same as previous proof. Use uncertainty principle for data completion.

$\square$

The End