DTU

IPwin2021

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A primer on PDE’s

Partial Differential Equations

The math of the physical world:

- Mechanics: Newton's laws of motion
- Quantum mechanics: Schrödinger's equations
- Electro-magnetism: Maxwell's system
- Fluid dynamics: Navier-Stokes
- Biology: Cell membranes, Pandemic modeling!

Comes in $1, \mathbf{2}, 3, \ldots$ dimensions.

Linear and non-linear models.

Elliptic, parabolic, hyperbolic.

In general for differential operator L

$$L(u) = 0$$

Agenda

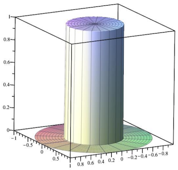
1. The Poisson equation and the Helmholtz equation
2. Integration by parts, Green's formulae and weak derivatives
3. Sobolev spaces and elliptic PDE's

References:
Walter Strauss, Introduction to Partial Differential Equations, second ed., Wiley 2008
Lawrence C. Evans, Partial Differential Equations, second ed., AMS 2010.

1. The Poisson equation and the Helmholtz equation

The Poisson equation

In the unit circle $\Omega = B(0,1) \subset \mathbb{R}^2$ consider



$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Notation

$$\Delta u = \nabla \cdot \nabla u$$

$$\Delta u = u_{xx} + u_{yy} \qquad \text{(Cartesian coordinates } (x,y))$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \qquad \text{(Polar coordinates}(r,\theta))$$

Models electric potential u and current $-\nabla u$ due to interior source f in a disk shaped conducting medium Ω.

The eigenvalue problem

Dirichlet Laplace

$$\begin{aligned} -\Delta v &= \lambda v \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Separation of polar variables $v(r,\theta) = R(r)\Theta(\theta)$ leads to Bessel's equation

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (r^2 - n^2)R = 0, \quad 0 < r < 1; \qquad (2)$$
$$\text{with } R(0) \text{ bounded and } R(1) = 0,$$

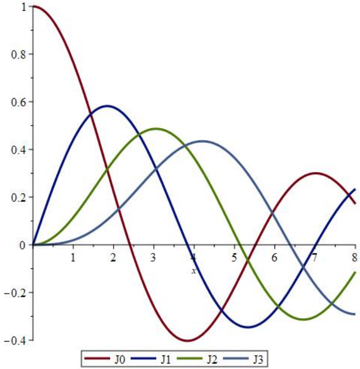
and

$$\Theta'' + n^2\Theta = 0 \qquad \text{in } \mathbb{R} \quad (+ \text{ periodic BC}).$$

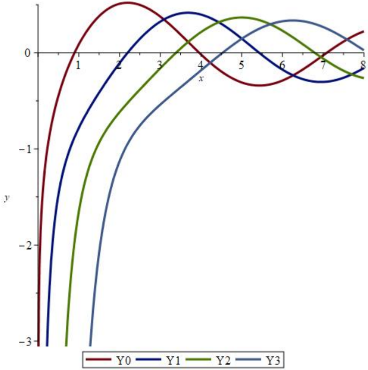
Bessel's equation of order n solved by Bessel functions of the first kind $J_n(\beta_{mn} r)$; β_{mn} is the m'th zero of J_n.

Bessel functions

First kind

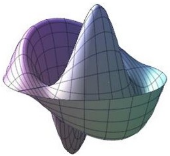


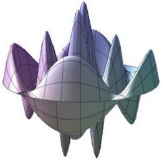
Second kind



The Dirichlet eigenfunctions and eigenvalues

$$v_{mn}(r,\theta) = c_{mn} J_n(\beta_{mn} r) e^{in\theta}; \quad \lambda_{mn} = \beta_{mn}^2.$$





Facts

- $\{v_{mn}\}_{m=1,n=0}^{\infty}$ constitute an orthonormal basis for $L^2(\Omega)$.
- Weyl's law (1911): The number of eigenvalues $N(\lambda)$ below the number λ is asymptotically

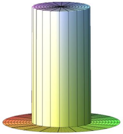
$$\frac{N(\lambda)}{\lambda} \sim 4$$

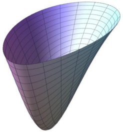
and consequently $\lambda_{mn} \to \infty$.

Solution to Poisson’s problem

We find for $-\Delta u = f \in L^2(\Omega)$ with

$$f = \sum_{m,n} f_{mn} v_{mn}$$



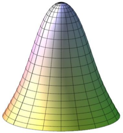


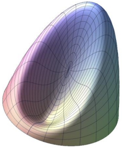
the unique solution u to the Poisson problem

$$u = \sum_{m,n} \frac{f_{mn}}{\lambda_{mn}} v_{mn}$$

with stability estimate

$$\|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$





Integral kernel - Green's function

Define for the Poisson problem the solution operator

$$\begin{aligned} K : L^2(\Omega) &\to L^2(\Omega) \\ f &\mapsto u \end{aligned}$$

This is an integral operator

$$u(y) = Kf(y) = \int_\Omega G(x, y) f(x) dx,$$

with G denoting the Dirichlet Green's function

$$-\Delta G(\cdot, y) = \delta_y \text{ in } \Omega, \qquad G(\cdot, y) = 0 \text{ on } \partial\Omega$$

formally represented by

$$G(x, y) = \sum_{m,n} \frac{1}{\lambda_{mn}} v_{mn}(y) v_{mn}(x).$$

The Helmholtz equation

For wave number k and $r = |x|$

$$\begin{aligned} \Delta u + k^2 u &= -f \text{ in } \mathbb{R}^2 \\ \lim_{r\to\infty} \sqrt{r}\,(\partial_r u - iku) &= 0 \qquad \text{(Sommerfeld radiation condition)} \end{aligned} \tag{3}$$

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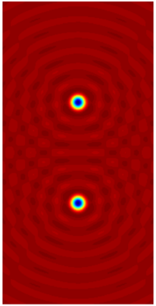

Hankel functions

$$H_n^{(1)} = J_n + iY_n; \quad H_n^{(2)} = J_n - iY_n$$

with Y_n are the Bessel function of second kind.

Unique solution to (3) given by volume potential

$$u(x) = -\int_{\mathbb{R}^2} H_0^{(1)}(|x-y|)f(y)dy.$$

Well-posed vs ill-posed

According to Hadamard a problem is call well-posed if

1. The problem has a solution (existence)
2. The solution is unique (uniqueness)
3. The solution depends continuously on the data (stability)

The problem (1) is well-posed.

If a condition fails, the problem is called ill-posed.

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Exercise 1
Consider the **inverse Poisson problem**: Given $u = Kf \in L^2(\Omega)$ from (1), find $f \in L^2(\Omega)$.

Is the problem well-posed?

2 Integration by parts, Green's formulae and weak derivatives

Fundamental theorem of calculus

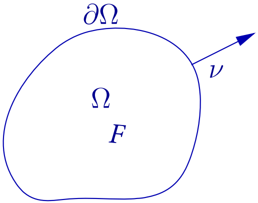
In one dimension

$$\int_a^b F'(x)dx = [F]_a^b = F(b) - F(a).$$

Integration by parts formula for $u, v \in C^1([a, b])$:

$$\int_a^b u'vdx = -\int_a^b uv'dx + [uv]_a^b$$

In higher dimensions



Divergence theorem

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} \nu \cdot F \, dS$$

Integration by parts:

$$\int_\Omega u_{x_j} v \, dx = -\int_\Omega u v_{x_j} dx + \int_{\partial\Omega} \nu_j u v \, dS.$$

or by stacking the partial derivatives

$$\int_\Omega \nabla u v \, dx = -\int_\Omega u \nabla v \, dx + \int_{\partial\Omega} \nu u v \, dS.$$

Weak derivative

If $v \in C_C^\infty(\overline{\Omega})$ the formula simply looks

$$\int_\Omega u_{x_j} v \, dx = -\int_\Omega u v_{x_j} \, dx$$

Definition:
Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Then we call $w \in L^1_{\mathrm{loc}}(\Omega)$ the weak derivative of u wrt. x_j provided that for all $v \in C_C^\infty(\Omega)$

$$\int_\Omega w v \, dx = -\int_\Omega u v_{x_j} \, dx.$$

Green's formulae

$$\int_{\Omega} u\Delta v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} \, dS \tag{4}$$

$$\int_{\Omega} u\Delta v - v\Delta u \, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \, dS \tag{5}$$

Green's formulae

$$\int_\Omega u\Delta v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, dS \tag{4}$$

$$\int_\Omega u\Delta v - v\Delta u \, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS \tag{5}$$

Apply (5) to u solving the Poisson problem and $v = G(\cdot, y)$ from

$$\begin{aligned} -\Delta G(\cdot, y) &= \delta_y \text{ in } \Omega, \\ G(\cdot, y) &= 0 \text{ on } \partial\Omega \end{aligned}$$

shows the formula from before

$$-u(y) + \int_\Omega G(x, y) f(x) dx = 0.$$

Fundamental solution and Green's functions

The fundamental solution of the Laplace equation $\Delta u = 0$ is in two dimensions

$$\Phi(x) = -\frac{1}{2\pi}\log|x|, \quad -\Delta\Phi(x-y) = \delta_y.$$

The Dirichlet Green's function satisfies the same PDE, so

$$G(x,y) = \Phi(x-y) + H_y(x)$$

with H being harmonic defined by

$$-\Delta H_y = 0, \qquad H_y(x) = -\Phi(x-y) \text{ for } x \in \partial\Omega.$$

Neumann function N_y satisfies

$$-\Delta N_y = \delta_y, \qquad \frac{\partial}{\partial\nu}N_y = -\frac{1}{|\partial\Omega|} \text{ on } \partial\Omega.$$

3 Sobolev spaces and elliptic PDE’s

Sobolev spaces

$$H^1(\Omega) = \left\{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\right\}.$$

Inner product:

$$(u, v)_{H^1(\Omega)} = \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx$$

and induced norm

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \left(|u|^2 + |\nabla u|^2\right)\, dx.$$

Sobolev spaces - Trace spaces

Restriction – the trace – to the boundary is well defined in $H^1(\Omega)$

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$
$$u \mapsto u|_{\partial\Omega}.$$

Boundary spaces:

$$\mathrm{Range}(T) = H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$$
$$H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$$

In particular we write

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\}.$$

Poincaré inequality for functions in $H_0^1(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Weak solution of the generalized Poisson equation

Consider

$$
\begin{aligned}
-\nabla \cdot A \nabla u &= f \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{6}
$$

Here $A : \Omega \to R^{nxn}$ is symmetric and satisfies for all $\xi \in \mathbb{R}^n$, $|\xi|_2 = 1$ (a.e. x)

$$
0 < a \leq \xi^T A(x) \xi \leq b < \infty. \qquad \text{(Coercivity/ellipticity)}
$$

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$$0 < a \leq \xi^T A(x)\xi \leq b < \infty. \qquad \text{(Coercivity/ellipticity)}$$

Multiply by function $v \in C_C^\infty(\Omega)$ and integrate by parts

$$\begin{aligned} -\int_\Omega \nabla \cdot A\nabla u v dx &= \int_\Omega A\nabla u \cdot \nabla v dx + \int_{\partial\Omega} (\nu \cdot A\nabla u) v \, dS \\ &= \int_\Omega f v dx. \end{aligned}$$

We call a function $u \in H_0^1(\Omega)$ a weak solution to (6) provided that for all $v \in H_0^1(\Omega)$

$$\int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega f v dx. \tag{7}$$

Lax-Milgram Theorem

In the Hilbert space H, Assume that the bilinear form $a : H \times H \mapsto \mathbb{R}$ satisfies for some $\alpha, \beta > 0$ and all $u, v \in H$

1. $|a(u, v)| \leq \alpha \|u\| \|v\|$,
2. $\beta \|u\|^2 \leq a(u, u)$.

Then for any functional $f \in H^*$ the problem

$$a(u, v) = f(v) \text{ for all } v \in H$$

has a unique solution $u \in H$ with $\|u\| \leq C\|f\|$.

Well-posedeness of Poisson problem

The Poisson equation has a unique weak solution in $H_0^1(\Omega)$
Proof: With $H = H_0^1(\Omega)$

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v \, dx, \quad f(v) = \int_\Omega fv \, dx$$

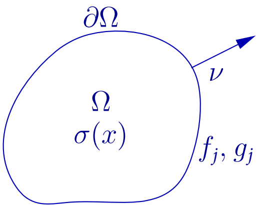
the assumptions from Lax-Milgram are satisfied: a is a bilinear form on $H_0^1(\Omega)$ and $f \in [H_0^1(\Omega)]^* = H^{-1}(\Omega)$.

1. $|a(u,v)| \leq b\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \leq b\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}$. (Cauchy-Schwarz)
2. $a\|u\|^2_{H_0^1(\Omega)} \leq aC\|\nabla u\|^2_{L^2(\Omega)} \leq C\int_\Omega A\nabla u \cdot \nabla u dx = Ca(u,u)$. (Poincaré + ellipticity)

Conductivity equation

A voltage potential $u \in H^1(\Omega)$ generated by boundary voltage $f \in H^{1/2}(\partial\Omega)$ satisfies conductivity equation

$$\begin{aligned} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f. \end{aligned}$$



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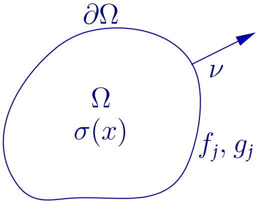
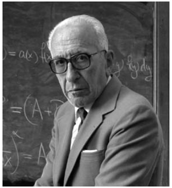


Exercise 2: Show that the above equation has a unique solution in $H^1(\Omega)$. (Hint: consider $\tilde{u} = u - \tilde{f}$ for some $\tilde{f} \in H^1(\Omega)$ with $\tilde{f}|_{\partial\Omega} = f$ and use the previous uniqueness result)

The Calderón problem

Measure electric normal current through at the boundary

$$g = \nu \cdot \sigma \nabla u|_{\partial\Omega}.$$



Dirichlet to Neumann (voltage to current) map

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$f \mapsto g$$

weakly defined by

$$\langle \Lambda_\sigma f, h \rangle = \int_{\partial\Omega} (\Lambda_\sigma f) h \, dS = \int_\Omega \sigma \nabla u \cdot \nabla v \, dx, \qquad \text{(any) } v|_{\partial\Omega} = h \in H^{1/2}(\partial\Omega).$$

Inverse problem (1980): How can we stably recover σ from knowledge of Λ_σ?

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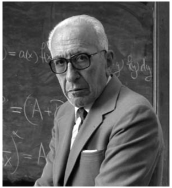

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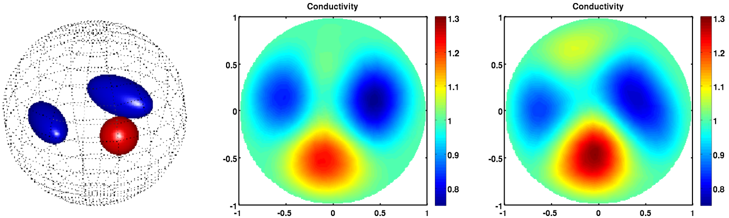
Exercise 3: Show that Λ_σ is well-defined in the weak form.

Electrical Impedance Tomography



Thanks for your attention