Non-scattering Wave Numbers

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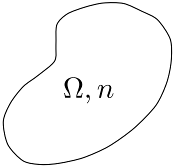
Joint with Jingni Xiao

Scattering



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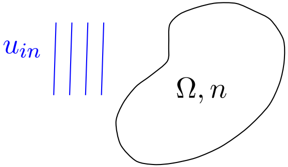

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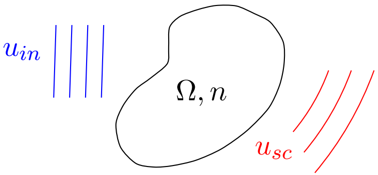

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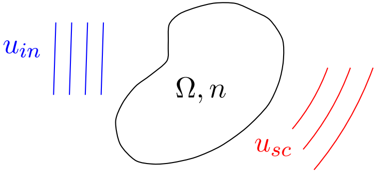

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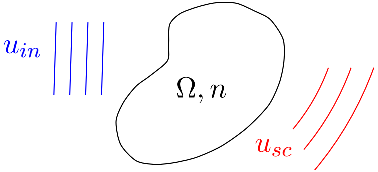

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$\Delta u + k^2 n u = 0$ in $\mathbb{R}^d$

$n = 1$ in Ω^c

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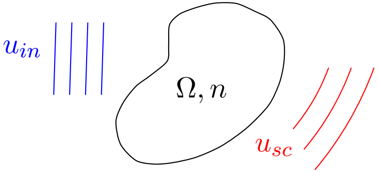

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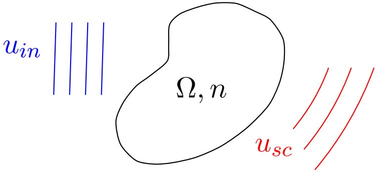

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$$\Delta u_{k,in} + k^2 u_{k,in} = 0 \quad \text{in } \mathbb{R}^d;$$

$u_{sc} = u_{k,sc}$: scattered wave, with Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial}{\partial r} u_{k,sc} - ik u_{k,sc} \right) = 0 \quad \text{uniformly for all directions.}$$

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— Notice: this (ITEP) is only a **necessary** condition for Non-scattering.

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- **Questions**:

$$\mathscr{A} = \mathscr{B}\ ? \qquad\qquad \#\mathscr{A} = \infty\ ?$$

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— Let us, as an example, consider the case $\Omega = B_1$ (ball of radius 1) in $\mathbb{R}^2$ with n_0 a constant.

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$$\begin{aligned} J_m(k\sqrt{n_0})A - J_m(k) &= 0 \\ \sqrt{n_0} J_m'(k\sqrt{n_0})A - J_m'(k) &= 0 \end{aligned}$$

has a solution.

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$$d_m\left(\frac{2j\pi - \frac{\pi}{2}}{1-\sqrt{n_0}}\right) > 0 \quad \text{and} \quad d_m\left(\frac{2j\pi + \frac{\pi}{2}}{1-\sqrt{n_0}}\right) < 0$$

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We note that

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\frac{1}{2\pi}\int_{S^1} e^{-im\theta_\xi} e^{ikx\cdot\xi}\, d\xi &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta} e^{ik|x|\cos(\theta-\theta_x)}\, d\theta \\
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So what happens if you perturb the disk to an ellipse?

Theorem (Herglotz Waves [Vogelius-Xiao '20])

Let $\phi \in C^\infty(\mathbb{S}^1)\backslash\{0\}$ *and* $n_0 \in \mathbb{R}_+\backslash\{1\}$*. Let* Ω *denote the ellipse*

$$\Omega = \Omega_{A,B} = \{(x_1, x_2) \ : \ \frac{x_1^2}{A^2} + \frac{x_2^2}{B^2} < 1 \ \} \ .$$

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There exists a positive constant ϵ *(independent of* ϕ*), so that if* $0 < |\frac{B}{A} - 1| < \epsilon$ *then one can find at most finitely many positive wave numbers* $k = k_j, j = 1, 2, \ldots, N$, *(possibly depending on* ϕ*) for which the problem*

$$\begin{cases} \Delta u_{k,sc} + k^2 n_0 u_{k,sc} = k^2(1 - n_0)u_{k,in} & \text{in } \Omega, \\ u_{k,sc} = \partial_\nu u_{k,sc} = 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution $u_{k,sc}$, *with*

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Cont'd

- Consequence: At most finitely many non-scattering wave numbers for $(\Omega, n_0, \mathbf{H}[k,\phi])$.
- Some generalizations ([Vogelius-Xiao, in preparation]):
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What happens for other incident waves, e.g., plane waves $e^{ikx\cdot\xi}$?

- Suppose Ω is $C^{1,\alpha}$, and suppose there exists a non-scattering wave number for $(\Omega, n_0, e^{ikx\cdot\xi})$. Then Ω must be real analytic (in other words its boundary is a real analytic hypersurface).
- If Ω is a ball then there are exists no non-scattering wave number for $(\Omega, n_0, e^{ikx\cdot\xi})$.
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- This raises a natural question: if Ω is real analytic but not a ball what might one say about the number of non-scattering wave numbers for $(\Omega, n_0, e^{ikx\cdot\xi})$

Theorem (Plane Waves [Vogelius-Xiao '20])

Let Ω be a bounded, strictly convex $C^{2,\alpha}$ domain in $\mathbb{R}^2$, and suppose $n_0 \in \mathbb{R}_+ \backslash \{1\}$. Given $u_{k,in} = e^{ikx\cdot\xi}$, for a fixed direction $\xi \in \mathbb{S}^1$, there exist at most finitely many positive wave numbers $k = k_j$, $j = 1, \ldots, N$, such that

$$\begin{cases} \Delta u_{k,sc} + k^2 n_0 u_{k,sc} = k^2(1-n_0)u_{k,in} & \text{in } \Omega, \\ u_{k,sc} = \partial_\nu u_{k,sc} = 0 & \text{on } \partial\Omega, \end{cases}$$

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We may in particular take $w = e^{ik\sqrt{n_0}\,\eta\cdot x}$ for any $\eta \in \mathbb{S}^1$

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- As a result we obtain a contradiction when (1) $u_{k,in}$ is a plane wave or (2) when $u_{k,in}$ is a Herglotz wave and Ω is a proper ellipse.

Connection with Schiffer's conjecture

Ω always scatters if:

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Conjecture: The only simply connected domains in $\mathbb{R}^d$ that fail to have the Schiffer Property are balls.

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Note that: balls have infinitely many Neumann eigenvalues with corresponding eigenfunctions of constant Dirichlet boundary trace.

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- Any Lipschitz domain Ω that fails to have the Schiffer Property, is real analytic.
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- If Ω is a simply connected Lipschitz domain with infinitely many Neumann eigenvalues with corresponding eigenfunctions of constant Dirichlet trace, then Ω is a ball.
  [Berenstein-Yang '87, Vogelius '94]