Recent results concerning "small" change in boundary condiditions

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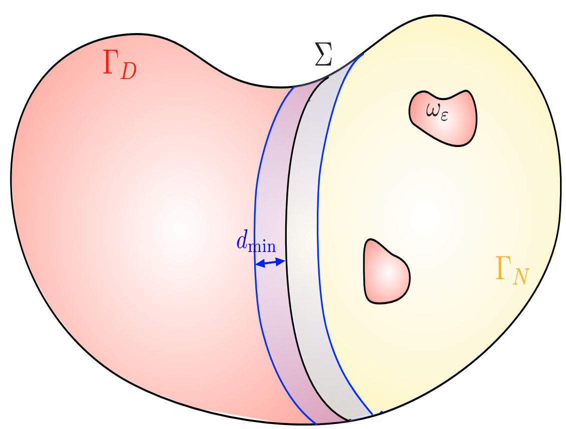

The perturbation settings

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega \\ \frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_N \\ u_0 = 0 & \text{on } \Gamma_D \end{cases} .$$

$$\begin{cases} \Delta u_\epsilon = f & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \omega_\epsilon \\ u_\epsilon = 0 & \text{on } \Gamma_D \cup \omega_\epsilon \end{cases} . \qquad \begin{cases} \Delta u_\epsilon = f & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \Gamma_N \cup \omega_\epsilon \\ u_\epsilon = 0 & \text{on } \Gamma_D \setminus \omega_\epsilon \end{cases} .$$

In the first case ω_ϵ is a subset of Γ_N, in the last case ω_ϵ is a subset of Γ_D.

GOAL: Find an asymptotic expression for $u_\epsilon(x) - u_0(x)$ that can be used to determine best location for ω_ϵ.

Assumptions: (1) $\omega_\epsilon \subset \partial\Omega$ consists of a finite number of connected, open Lipschitz subdomains, the closures of which do not intersect. (2) ω_ϵ lies either strictly inside Γ_N, or stricly inside Γ_D.

General Representation Formula

For $\omega_\epsilon \subset \Gamma_N$, when we impose $u_\epsilon = 0$ on ω_ϵ:

$$u_\epsilon(x) = u_0(x) - \mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y) N(x,y) d\mu(y) + o(\mathrm{cap}_D(\omega_\epsilon))$$

for $x \in K \subset\subset \Omega$.

$N(x,y)$ is a fundamental solution associated with Δ:

$$-\Delta_y N(x,y) = \delta_x \text{ in } \Omega$$

$$\frac{\partial N}{\partial n_y} = 0 \text{ on } \Gamma_N \ , \quad N(x,\cdot) = 0 \text{ on } \Gamma_D \ .$$

The capacity $\mathrm{cap}_D(\omega_\epsilon)$ is defined by

$$\mathrm{cap}_D(\omega_\epsilon) = \min\left\{ \int_{\mathbb{R}^d} \left(|\nabla v|^2 + |v|^2 \right) \, dx \ : \ v \in H^1(\mathbb{R}^d) \ , \ v = 1 \text{ on } \omega_\epsilon \right\}$$

About $\mathrm{cap}_D(\omega_\epsilon)$

Define χ_ϵ by

$$\begin{aligned} &\Delta \chi_\epsilon = 0 \ \text{ in } \Omega \ , \\ &\frac{\partial \chi_\epsilon}{\partial n} = 0 \ \text{ on } \Gamma_N \setminus \omega_\epsilon \ , \\ &\chi_\epsilon = 0 \ \text{ on } \Gamma_D \ \text{ and } \chi_\epsilon = 1 \ \text{ on } \omega_\epsilon \ . \end{aligned}$$

Then

$$c \int_\Omega |\nabla \chi_\epsilon|^2 \, dx \leq \ \mathrm{cap}_D(\omega_\epsilon) \leq C \int_\Omega |\nabla \chi_\epsilon|^2 \, dx$$

$$\text{and} \quad \int_\Omega |\chi_\epsilon|^2 \, dx \leq C \ \mathrm{cap}_D(\omega_\epsilon)^{3/2}$$

We calculate

$$\frac{1}{\mathrm{cap}_D(\omega_\epsilon)}\int_{\partial\Omega}\frac{\partial\chi_\epsilon}{\partial n}\chi_\epsilon\,\phi \qquad = \frac{1}{\mathrm{cap}_D(\omega_\epsilon)}\int_\Omega \nabla\chi_\epsilon\cdot\nabla(\chi_\epsilon\phi)$$
$$= \frac{1}{\mathrm{cap}_D(\omega_\epsilon)}\int_\Omega |\nabla\chi_\epsilon|^2\phi + O(\|\phi\|_{C^1}\mathrm{cap}_D(\omega_\epsilon)^{1/4})$$

and so

$$\frac{1}{\mathrm{cap}_D(\omega_\epsilon)}\frac{\partial\chi_\epsilon}{\partial n}\chi_\epsilon \quad \text{converges weak* in } (C^1(\partial\Omega))' \text{ to some } \mu$$

(after extraction of a sub-sequence)

furthermore $\quad |\mu(\phi)| \leq C\|\phi\|_{C^0(\partial\Omega)}$ for all $\phi\in C^1(\partial\Omega)$,

in other words μ is a positive Radon measure

$$\mu(\phi) = \int_{\partial\Omega}\phi d\mu$$

Note: the support of μ lies inside any compact set, which contains all ω_ϵ (from a certain point in the sub-sequence).

Let $r_\epsilon = u_\epsilon - u_0$. This satisfies the estimates

$$\|r_\epsilon\|_{H^1(\Omega)} \leq C\,\mathrm{cap}_D(\omega_\epsilon)^{1/2} \quad \text{and} \quad \|r_\epsilon\|_{L^2(\Omega)} \leq C\,\mathrm{cap}_D(\omega_\epsilon)^{3/4} \ .$$

Suppose ψ vanishes on Γ_D, then

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n}\,\psi &= \int_{\Omega} \nabla r_\epsilon \cdot \nabla(\chi_\epsilon \psi) \\
&= \int_{\Omega} \nabla r_\epsilon \cdot \nabla \chi_\epsilon\,\psi + O(\mathrm{cap}_D(\omega_\epsilon)^{5/4}) \\
&= \int_{\partial\Omega} r_\epsilon \frac{\partial \chi_\epsilon}{\partial n}\,\psi + O(\mathrm{cap}_D(\omega_\epsilon)^{5/4}) \\
&= \int_{\partial\Omega} r_\epsilon \frac{\partial \chi_\epsilon}{\partial n} \chi_\epsilon\,\psi + O(\mathrm{cap}_D(\omega_\epsilon)^{5/4}) \\
&= -\int_{\partial\Omega} u_0 \frac{\partial \chi_\epsilon}{\partial n} \chi_\epsilon\,\psi + O(\mathrm{cap}_D(\omega_\epsilon)^{5/4}) \ .
\end{aligned}$$

With $\psi = N(x, \cdot)$:

$$\int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n_y} N(x,y)\, ds_y = -\int_{\partial\Omega} u_0(y) \frac{\partial \chi_\epsilon}{\partial n_y} \chi_\epsilon N(x,y)\, ds_y + O(\mathrm{cap}_D(\omega_\epsilon)^{5/4}) ,$$

and so

$$\lim \frac{1}{\mathrm{cap}_D(\omega_\epsilon)} \int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n_y} N(x,y)\, ds_y = -\int_{\partial\Omega} u_0(y)\, N(x,y)\, d\mu_y$$

We also calculate

$$\begin{aligned} u_\epsilon(x) - u_0(x) &= r_\epsilon(x) = -\int_\Omega r_\epsilon(y) \Delta_y N(x,y)\ dy \\ &= \int_\Omega \nabla r_\epsilon \cdot \nabla_y N(x,y)\ dy \\ &= \int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n} N(x,y)\, ds_y\ . \end{aligned}$$

Altogether

$$\lim \frac{1}{\mathrm{cap}_D(\omega_\epsilon)}(u_\epsilon(x) - u_0(x)) = -\int_{\partial\Omega} u_0(y)\, N(x,y)\, d\mu_y \ ,$$

or

$$u_\epsilon(x) = u_0(x) - \mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y)\, N(x,y)\, d\mu_y + o(\mathrm{cap}_D(\omega_\epsilon)) \ .$$

Immediate application:

$$\begin{aligned}
\int_\Omega u_\epsilon\, f\ dx \quad &= \quad \int_\Omega u_0\, f\ dx \\
&\qquad -\mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y) \int_\Omega N(x,y)\, f(x)\, dx\, d\mu_y \\
&= \quad \int_\Omega u_0\, f\ dx \\
&\qquad -\mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y)^2\, d\mu_y \le \int_\Omega u_0\, f\ dx \ ,
\end{aligned}$$

in other words: the compliance asymptotically (strictly) decreases by introducing a small “clamped” area inside the “free” boundary – by how much depends on the size and shape of ω_ϵ and on u_0^2.

We have a similar asymptotic formula for the insertion of homogeneous Neumann boundary conditions on ω_ϵ inside Γ_D

$$u_\epsilon(x) = u_0(x) + e(\omega_\epsilon) \int_{\partial\Omega} \frac{\partial u_0}{\partial n}(y) \, \frac{\partial N}{\partial n_y}(x, y) \, d\nu_y + o(e(\omega_\epsilon)) \ .$$

$e(\omega_\epsilon)$ (you may call it the Neumann capacity) and the Radon measure ν are defined in a way that parallels the case before:

Let $\kappa \in C_c^\infty(\mathbb{R}^d \setminus \omega_\epsilon)$ with $\kappa = \pm 1$ on ω_ϵ ω_ϵ has only finitely connected components, i.e.), there are only finitely many choices of ± 1. Define z_κ by

$$\begin{aligned} -\Delta z_\kappa + z_\kappa &= 0 \quad \text{in } \mathbb{R}^d \setminus \omega_\epsilon \ , \\ \frac{\partial z_\kappa}{\partial n} &= \kappa \quad \text{on } \omega_\epsilon \ . \end{aligned}$$

Then

$$e(\omega_\epsilon) = \max_\kappa \left\{ \int_{\mathbb{R}^d \setminus \omega_\epsilon} (|\nabla z_\kappa|^2 + z_\kappa^2) \right\} \ .$$

Define ζ_ϵ by

$$\begin{aligned} &\Delta \zeta_\epsilon = 0 \quad \text{in } \Omega \ , \\ &\frac{\partial \zeta_\epsilon}{\partial n} = 0 \quad \text{on } \Gamma_N \text{ and } \frac{\partial \zeta_\epsilon}{\partial n} = 1 \quad \text{on } \omega_\epsilon \ , \\ &\zeta_\epsilon = 0 \quad \text{on } \Gamma_D \setminus \omega_\epsilon \ . \end{aligned}$$

Then

$$c \int_\Omega |\nabla \zeta_\epsilon|^2 \ dx \leq e(\omega_\epsilon) \leq C \int_\Omega |\nabla \zeta_\epsilon|^2 \ dx$$

and $\quad \int_\Omega |\zeta_\epsilon|^2 \; dx \leq C e(\omega_\epsilon)^{3/2}$

Similar to before the positive Radon measure ν is obtained as the weak* limit of

$$\frac{1}{e(\omega_\epsilon)} \frac{\partial \zeta_\epsilon}{\partial n} \zeta_\epsilon \; .$$

Immediate application:

$$\begin{aligned}
\int_\Omega u_\epsilon \, f \; dx \quad &= \quad \int_\Omega u_0 \, f \; dx \\
&\qquad + e(\omega_\epsilon) \int_{\partial\Omega} \frac{\partial u_0}{\partial n}(y) \, \frac{\partial}{\partial n_y} \int_\Omega N(x,y) \, f(x) \, dx \, d\nu_y \\
&= \quad \int_\Omega u_0 \, f \; dx \\
&\qquad + e(\omega_\epsilon) \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n}(y) \right)^2 \; d\nu_y \geq \int_\Omega u_0 \, f \; dx \; ,
\end{aligned}$$

in other words: the compliance asymptotically (strictly) increases by introducing a small “free area” inside the “clamped” boundary – by how much depends on the size and shape of ω_ϵ and on $\left(\frac{\partial u_0}{\partial n}\right)^2$.

Some concrete examples:

Suppose ω_ϵ is a “surfacic” ball

$$\omega_\epsilon = \{y \ : \ |y - y_0| < \epsilon\} \cap \partial\Omega \quad \text{for some } y_0 \in \partial\Omega \ ,$$

Then

$$\mathrm{cap}_D(\omega_\epsilon) = \begin{cases} O(\frac{1}{|\log \epsilon|}) & \text{for } d = 2 \\ O(\epsilon) & \text{for } d = 3 \end{cases} .$$

and

$$e(\omega_\epsilon) = O(\epsilon^d) \ \ d = 2, 3 \ .$$

In this case the measures μ and ν must be point masses (with support at y_0)

We now have the asymptotic formulas

(1) For insertion of a homogeneous Dirichlet boundary condition on $\omega_\epsilon \subset \Gamma_N$:

$$u_\epsilon(x) = u_0(x) - \frac{\pi}{|\log(\epsilon)|} u_0(y_0) N(x, y_0) + o(\frac{1}{|\log(\epsilon)|}) \quad \text{for } d = 2 \ ,$$

and

$$u_\epsilon(x) = u_0(x) - 4\epsilon u_0(y_0) N(x, y_0) + o(\epsilon) \quad \text{for } d = 3 \ .$$

(2) For insertion of a homogeneous Neumann boundary condition on $\omega_\epsilon \subset \Gamma_D$:

$$u_\epsilon(x) = u_0(x) + a_d \epsilon^d \frac{\partial u_0}{\partial n}(y_0) \frac{\partial N}{\partial n_y}(x, y_0) + o(\epsilon^d) \ .$$

with $a_2 = \frac{\pi}{2}$, and $a_3 = \frac{1}{3}$.