Electromagnetic imaging – an applied analyst's perspective

Michael S. Vogelius
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ANSWER 1: – Yes, if A is **isotropic**, i.e., if $A(x) = \gamma(x)I$.

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ANSWER 2: – Only very partially (up to a "pullback" by a diffeomorphism) if A is **anisotropic**. Sylvester (1990), Lee-Uhlmann (1987)["positive"], Kohn-Vogelius, Tartar (1987)["negative"].

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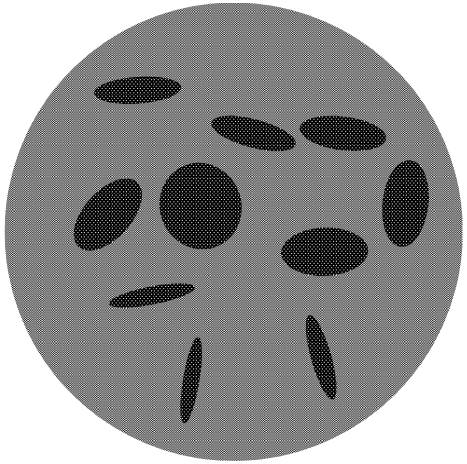
Collaborators: H. Ammari, E. Beretta, M. Brühl, Y. Capdeboscq, E. Francini, A.Friedman, M. Hanke, D. Hansen, S. Moskow, D. Volkov ....



$$\gamma_\epsilon(x) = \begin{cases} \gamma_0(x) & \text{in } \Omega \setminus \omega_\epsilon \\ \gamma_1(x) & \text{in } \omega_\epsilon \end{cases}$$

$$\omega_\epsilon \subset K_0 \subset \Omega$$

$$0 < \gamma_0(x), \gamma_1(x) < \infty$$

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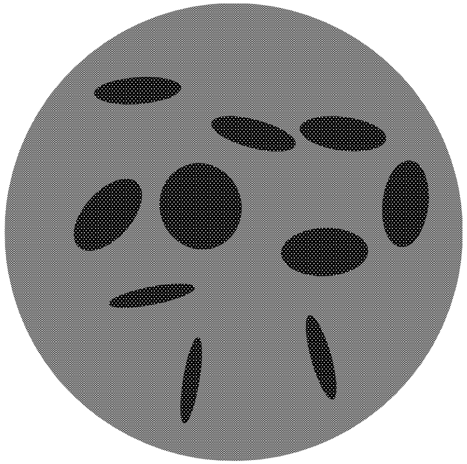

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Goal: Find an asymptotic expression for $(u_\epsilon - u_0)\big|_{\partial\Omega}$ that can be used to determine ω_ϵ (for $|\omega_\epsilon|$ small).

General Representation Formula

After the extraction of a subsequence:

$$\forall y \in \partial\Omega,\ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega \qquad\qquad\qquad\qquad + o(|\omega_\epsilon|)$$

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$N(x,y)$ is the Neumann function for $\nabla \cdot (\gamma_0 \nabla\ )$:

$$\nabla_x \cdot \big(\gamma_0 \nabla_x N(x,y)\big) = \delta_y \text{ in } \Omega$$

$$\gamma_0(x) \tfrac{\partial N}{\partial \nu_x} = \tfrac{1}{|\partial\Omega|} \text{ on } \partial\Omega.$$

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The probability measure $\mu = \lim\limits_{|\omega_\epsilon| \to 0} \dfrac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon}$ converges weak* in the dual of $C^0\left(\overline{\Omega}\right)$.

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$$\forall y \in \partial\Omega,\ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x,y) d\mu(x) + o(|\omega_\epsilon|)$$

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M is a matrix valued function in $L^2(\Omega, d\mu)$. The values of M are symmetric, positive definite matrices.

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$$\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \nabla u_\epsilon dx \rightharpoonup M \nabla u_0 d\mu$$

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$$\text{and } \left\| \nabla (u_\epsilon - u_0) \right\|_{L^2(\Omega)} \leq C |\omega_\epsilon|^{\frac{1}{2}} \quad (u_0 \text{ is smooth}).$$

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But, notice that $v_0^j = x_j + cst$ is not a solution to

$$\begin{cases} \nabla \cdot (\gamma_0 \nabla v_0) = 0 & \text{in } \Omega \\ \gamma_0 \frac{\partial v_0}{\partial \nu} = \psi & \text{on } \partial\Omega. \end{cases}$$

unless γ_0 is a constant!

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Instead, construct

$$\begin{cases} \nabla \cdot \left(\gamma_\epsilon \nabla v_\epsilon^j\right) = \nabla \cdot \left(\gamma_0 \nabla v_0^j\right) & \text{in } \Omega \\ \gamma_\epsilon \frac{\partial v_\epsilon^j}{\partial \nu} = \gamma_0 \frac{\partial v_0^j}{\partial \nu} & \text{on } \partial\Omega, \\ v_0^j = x_j + \text{cst}, & \end{cases}$$

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and define

$$\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial v_\epsilon^j}{\partial x_k} \xrightharpoonup{\text{def}} M_{jk} d\mu.$$

$$\text{To prove } \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} dx \rightharpoonup M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$$

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$$||$$

$$\tfrac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx$$

$$\text{To prove } \frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} dx \rightharpoonup M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$$

$$\frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla (u_\epsilon - u_0) \nabla v_\epsilon^j dx \quad = \quad \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla (u_\epsilon - u_0) \nabla v_0^j dx$$

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$$\downarrow$$

$$\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} d\mu$$

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$$\begin{array}{ccc}
\frac{1}{|\omega_\epsilon|} \int \gamma_\epsilon \nabla(u_\epsilon - u_0) \nabla v_\epsilon^j dx & = & \frac{1}{|\omega_\epsilon|} \int \gamma_0 \nabla(u_\epsilon - u_0) \nabla v_0^j dx \\
\| & & \| \\
\frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_0 \nabla v_\epsilon^j dx & & \frac{1}{|\omega_\epsilon|} \int (\gamma_0 - \gamma_\epsilon) \nabla u_\epsilon \nabla v_0^j dx \\
\downarrow & & \downarrow \\
\displaystyle\int (\gamma_0 - \gamma_1) M_{jk} \frac{\partial u_0}{\partial x_k} d\mu & & \displaystyle\int (\gamma_0 - \gamma_1) \lim_{\epsilon \to 0} \left(\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \frac{\partial u_\epsilon}{\partial x_j} \right) dx
\end{array}$$

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+compensated compactness (judicious integration by parts)

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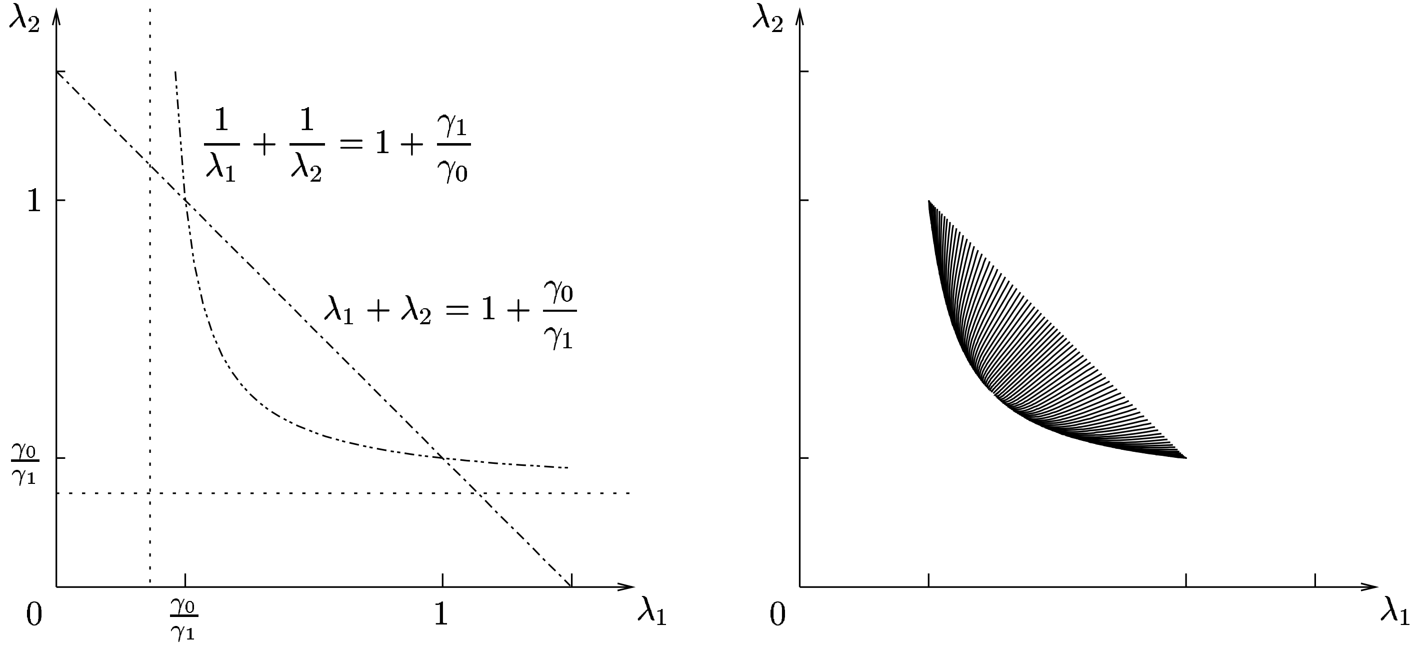
+compensated compactness (judicious integration by parts)

+ cut-offs.

Characterization of M



M is symmetric positive definite, and

$$\mathrm{Trace}\,(M) \quad \leq \quad n - 1 + \frac{\gamma_0}{\gamma_1}$$

$$\mathrm{Trace}\left(M^{-1}\right) \quad \leq \quad n - 1 + \frac{\gamma_1}{\gamma_0}$$

Bounds

M satisfies

$$\min\left(1, \frac{\gamma_0}{\gamma_1}\right) I_n \leq M \leq \max\left(1, \frac{\gamma_0}{\gamma_1}\right) I_n,$$

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$$n\ h\left(1, \ldots, 1, \frac{\gamma_0}{\gamma_1}\right) \leq \text{trace(M)} \leq n\ a\left(1, \ldots, 1, \frac{\gamma_0}{\gamma_1}\right).$$

h is the harmonic average, and a is the arithmetic average.

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Three of these bounds are attained for a single "sheet-like" inclusion. In that case the polarization egenvalues "parallel" to the sheet are 1, and the eigenvalue across the sheet is $\frac{\gamma_0}{\gamma_1}$. The fourth bound is attained for a single inclusion in the shape of a ball.

Applications

$$\forall y \in \partial\Omega,\ u_\epsilon(y) - u_0(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x) + o(|\omega_\epsilon|)$$

May be used :

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May be used :

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).

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May be used :

1. To detect location of diametrically small inhomogeneities (Brühl, Hanke, MV).

2. To estimate the volume of inhomogeneities of moderate size (Capdeboscq, MV)

Detecting locations

Suppose $\omega_\epsilon = \cup_{j=1}^{p}(z_j + \epsilon B_j)$ (the inhomogeneities "shrink" to points z_j).

$$
\begin{aligned}
D(\phi)(\cdot) &= |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, \cdot) d\mu(x) \\
&= |\omega_\epsilon| \sum_{j=1}^{p} (\gamma_1 - \gamma_0) \alpha_j M^j \nabla u_0(z_j) \cdot \nabla_x N(z_j, \cdot)
\end{aligned}
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Is linear in in ϕ (the prescribed boundary condition), its range is finite dimensional (of dimension np).

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Is linear in in ϕ (the prescribed boundary condition), its range is finite dimensional (of dimension np). In fact,

$$
\mathcal{R}(D) = \text{span}\{e_k \cdot \nabla_x N(z_j, \cdot)|_{\partial B} : k = 1, \ldots, n, j = 1, \ldots, p\}.
$$

Probe with $g_{z,d} = d \cdot \nabla_x N(z, \cdot)|_{\partial B}$. Then $g_{z,d} \in \mathcal{R}(D)$ iff $z \in \{z_j : j = 1, \ldots, p\}$. Note also that $\mathcal{R}(D)$ is well approximated by $\mathcal{R}(\Lambda_\epsilon - \Lambda_0)$ (the measured Neumann-Dirichlet "difference" map).

Detecting locations

Method:

Detecting locations

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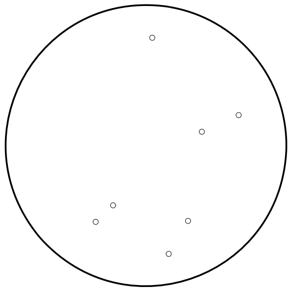
2. For a test point z, compute

$$\cot\,\theta_m(z) = \frac{\|P_m g_{z,d}\|}{\big\|(I - P_m)g_{z,d}\big\|}.$$

For $m = pn$, $z \in \{z_j : j = 1, \ldots, p\} \Leftrightarrow \cot\,\theta_m(z) = \infty$.

Detecting locations



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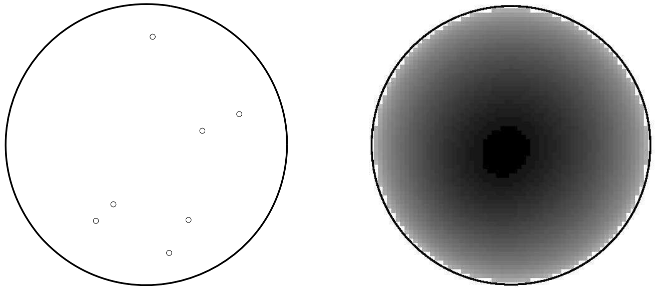

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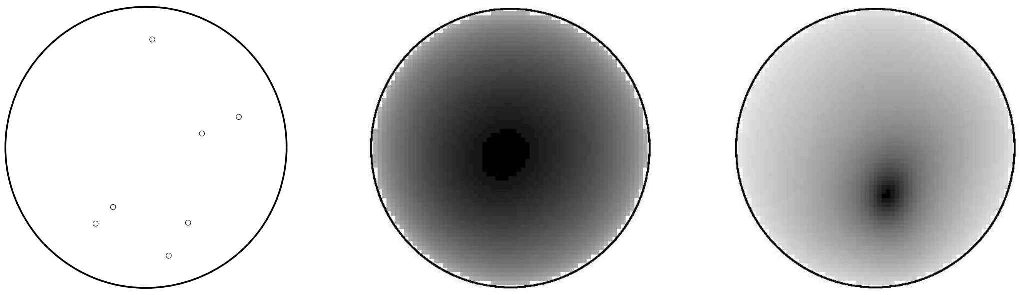

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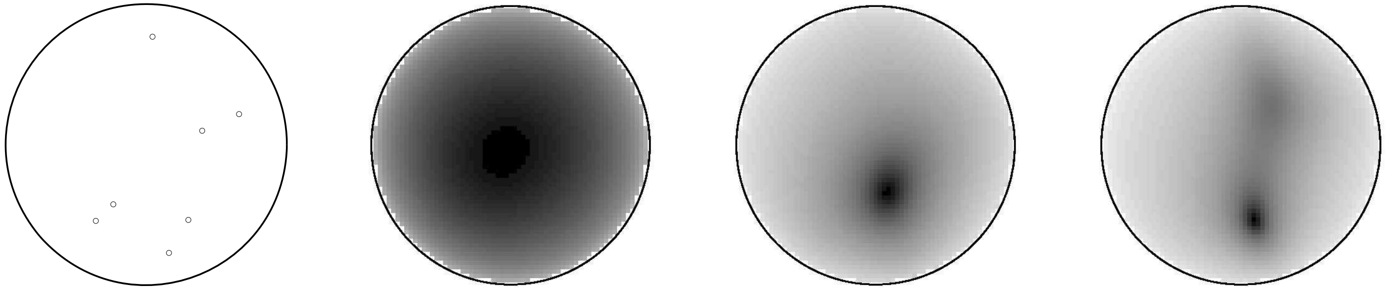

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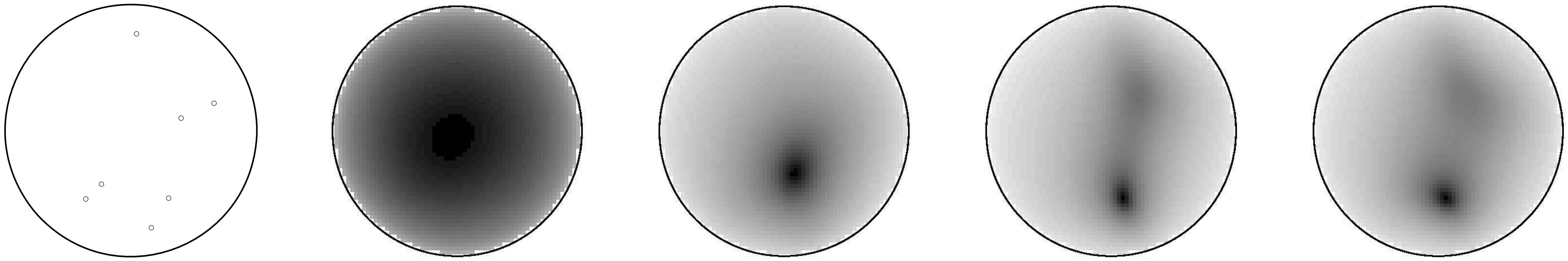

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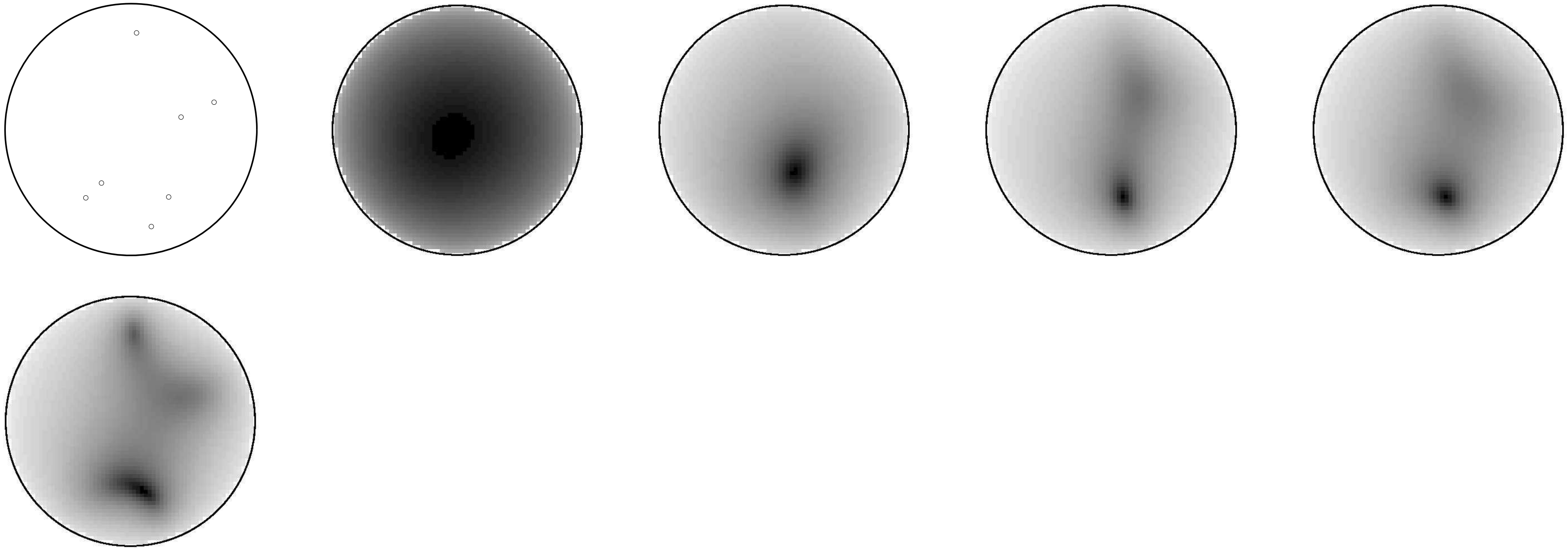

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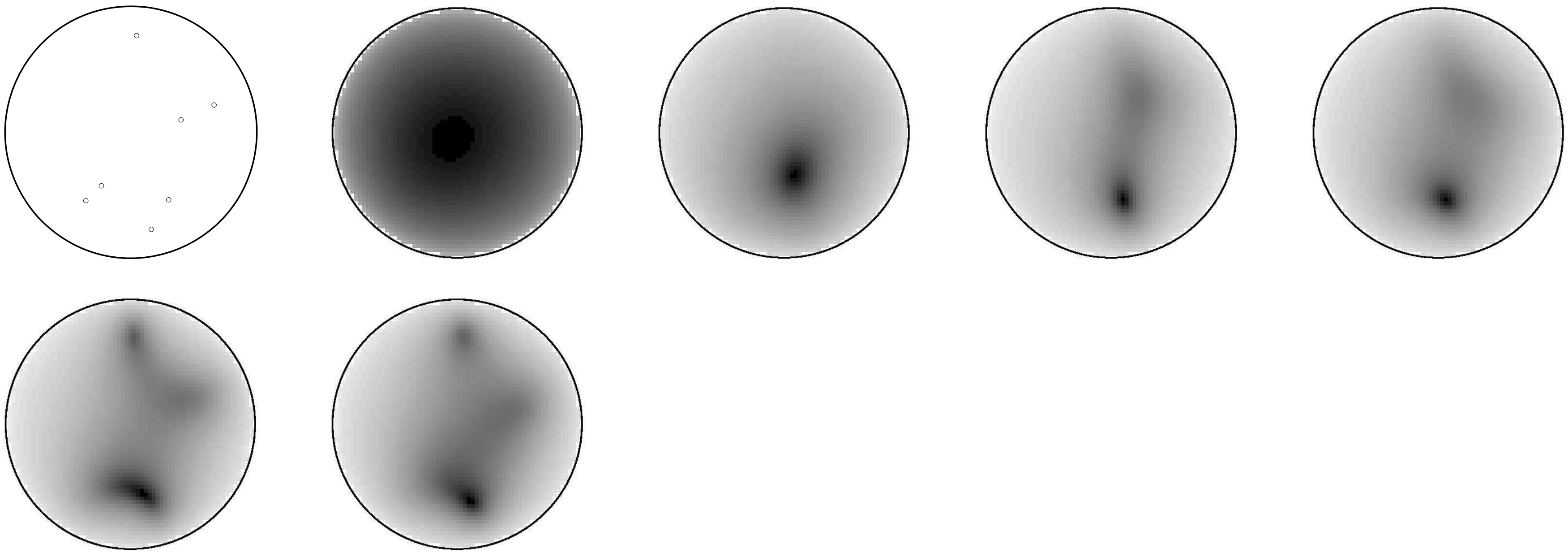

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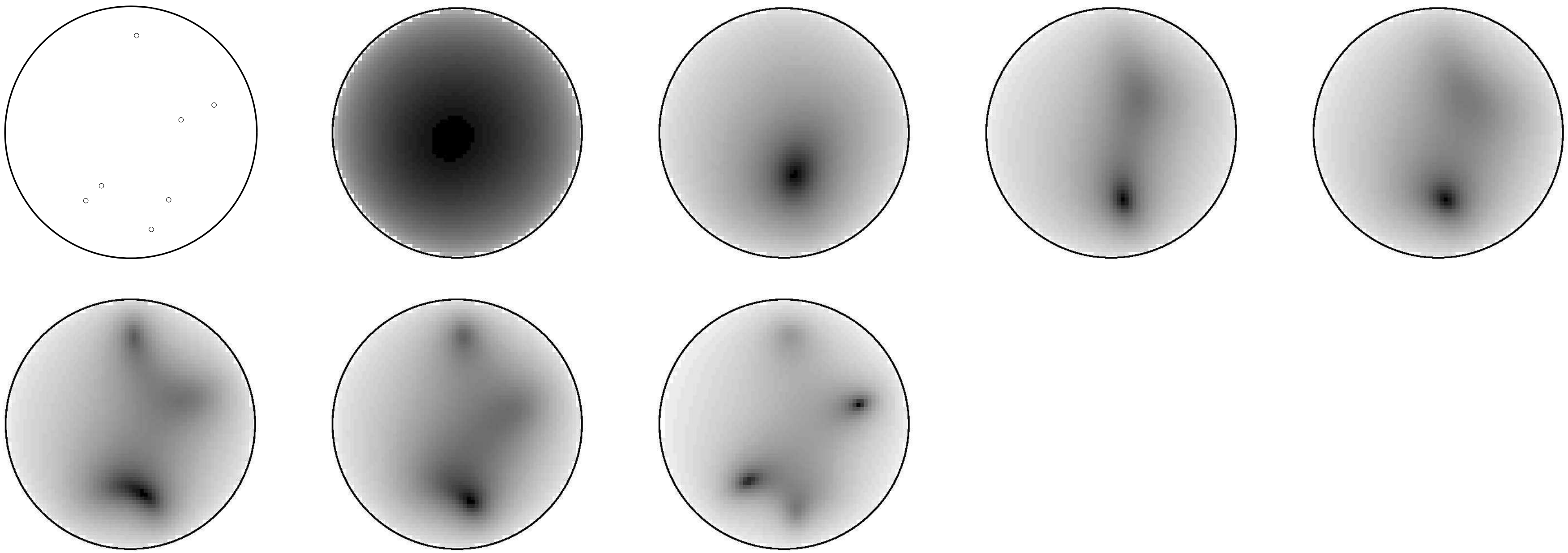

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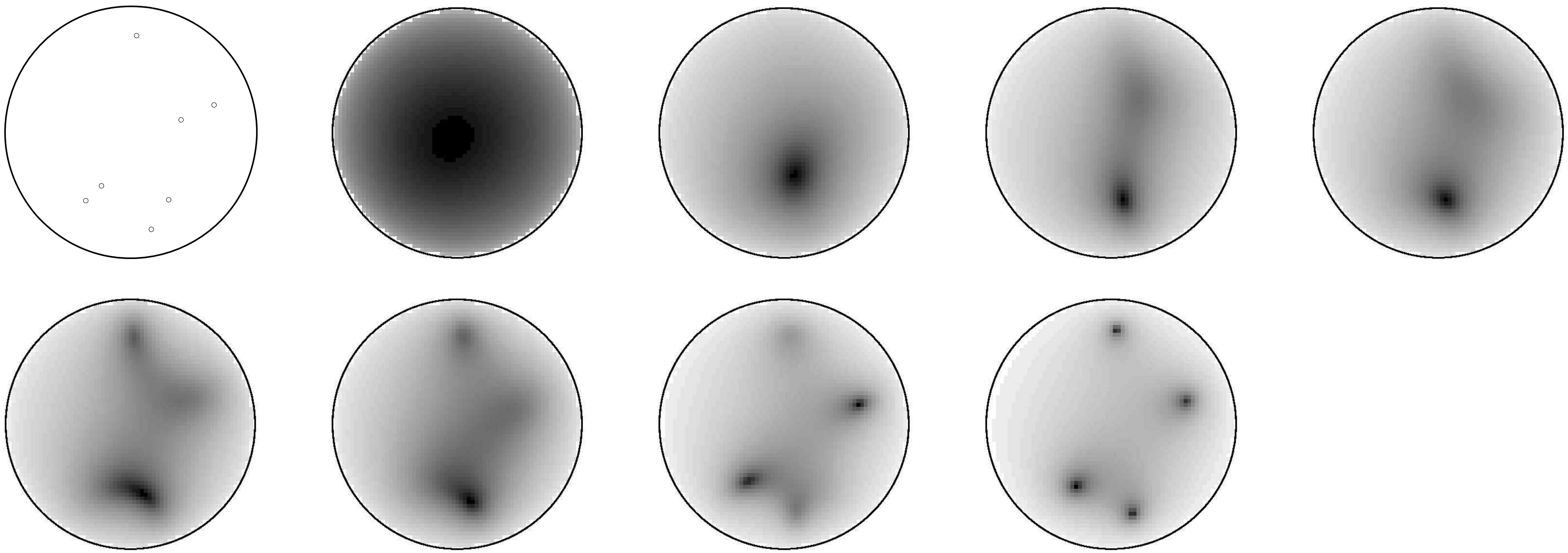

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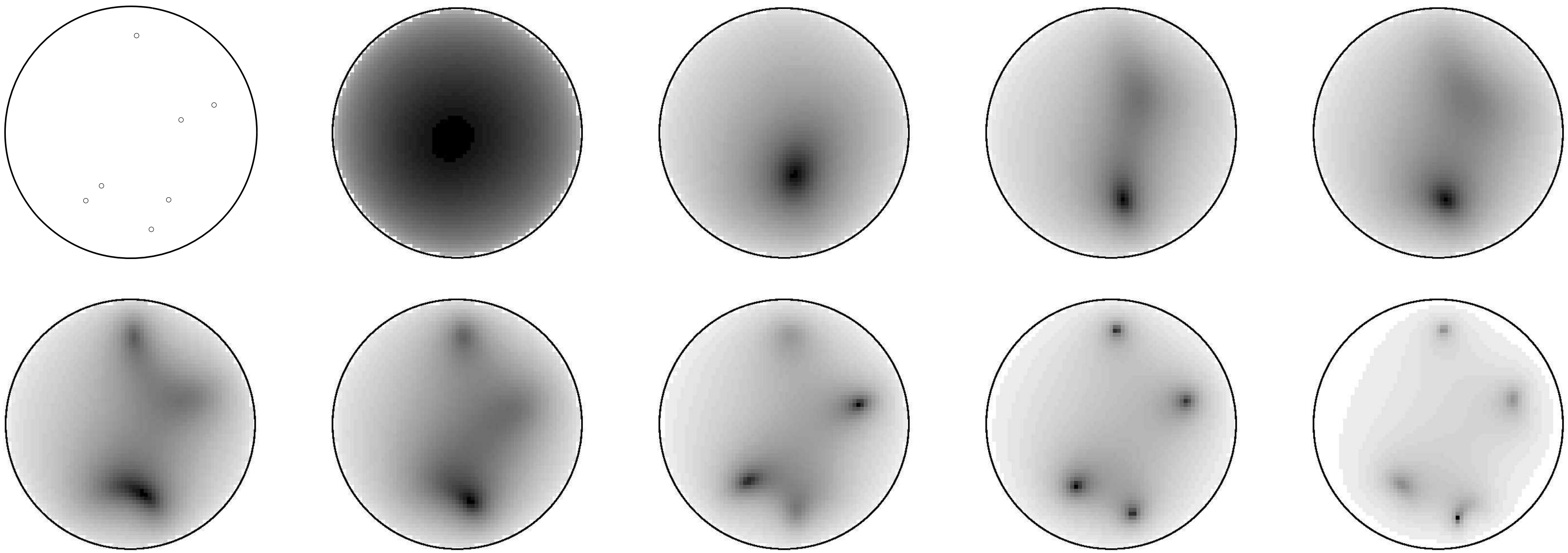

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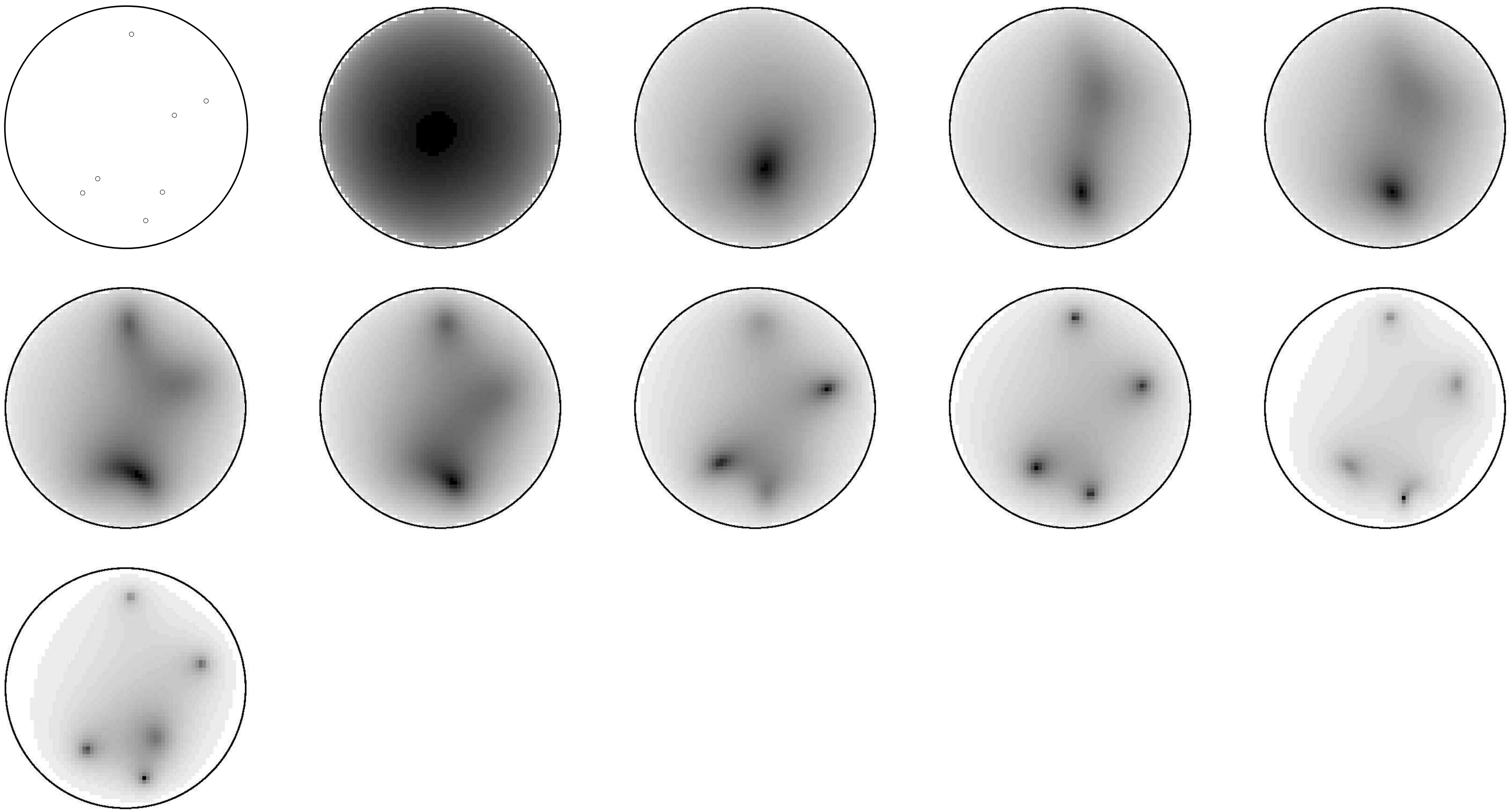

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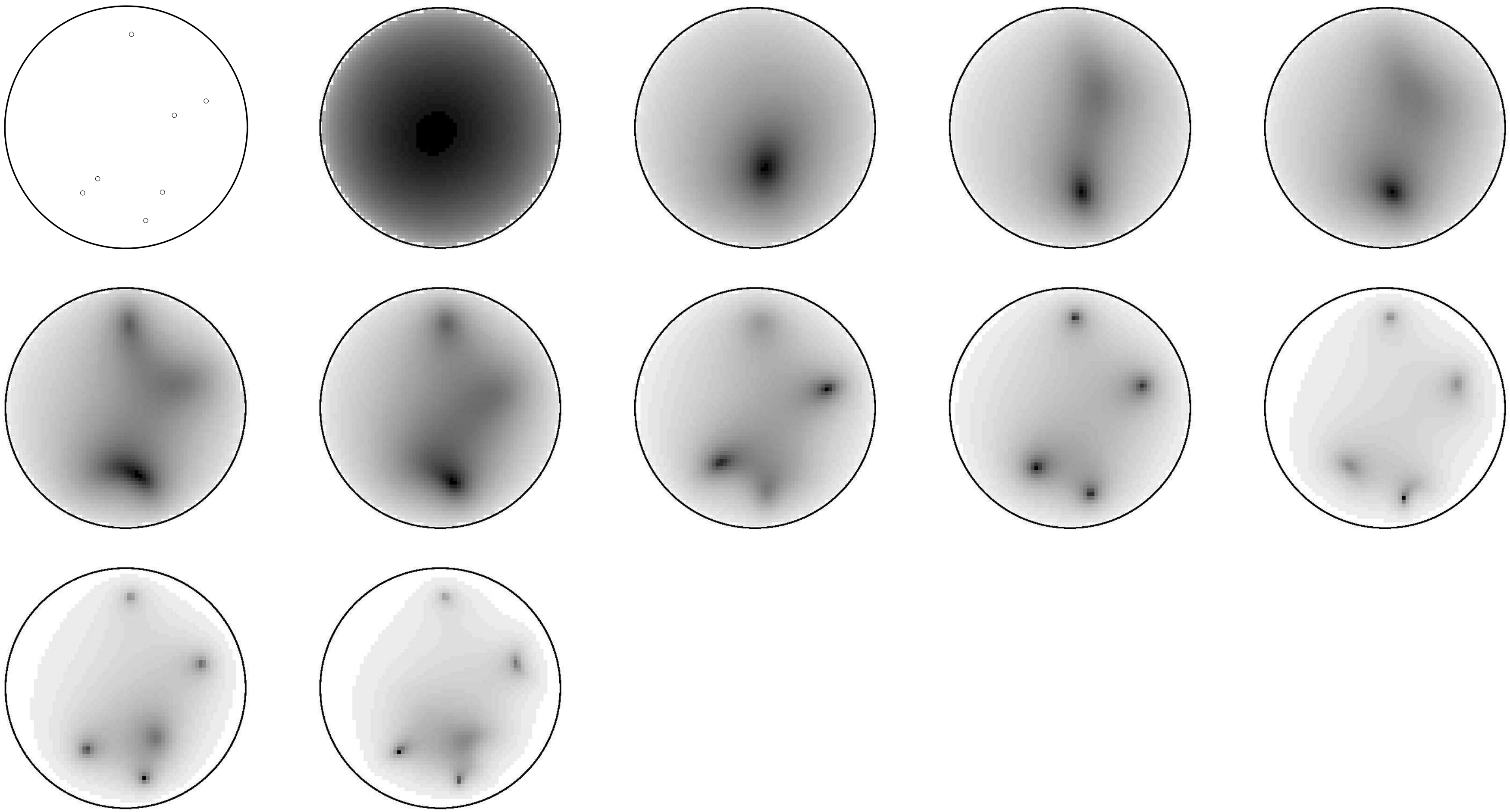

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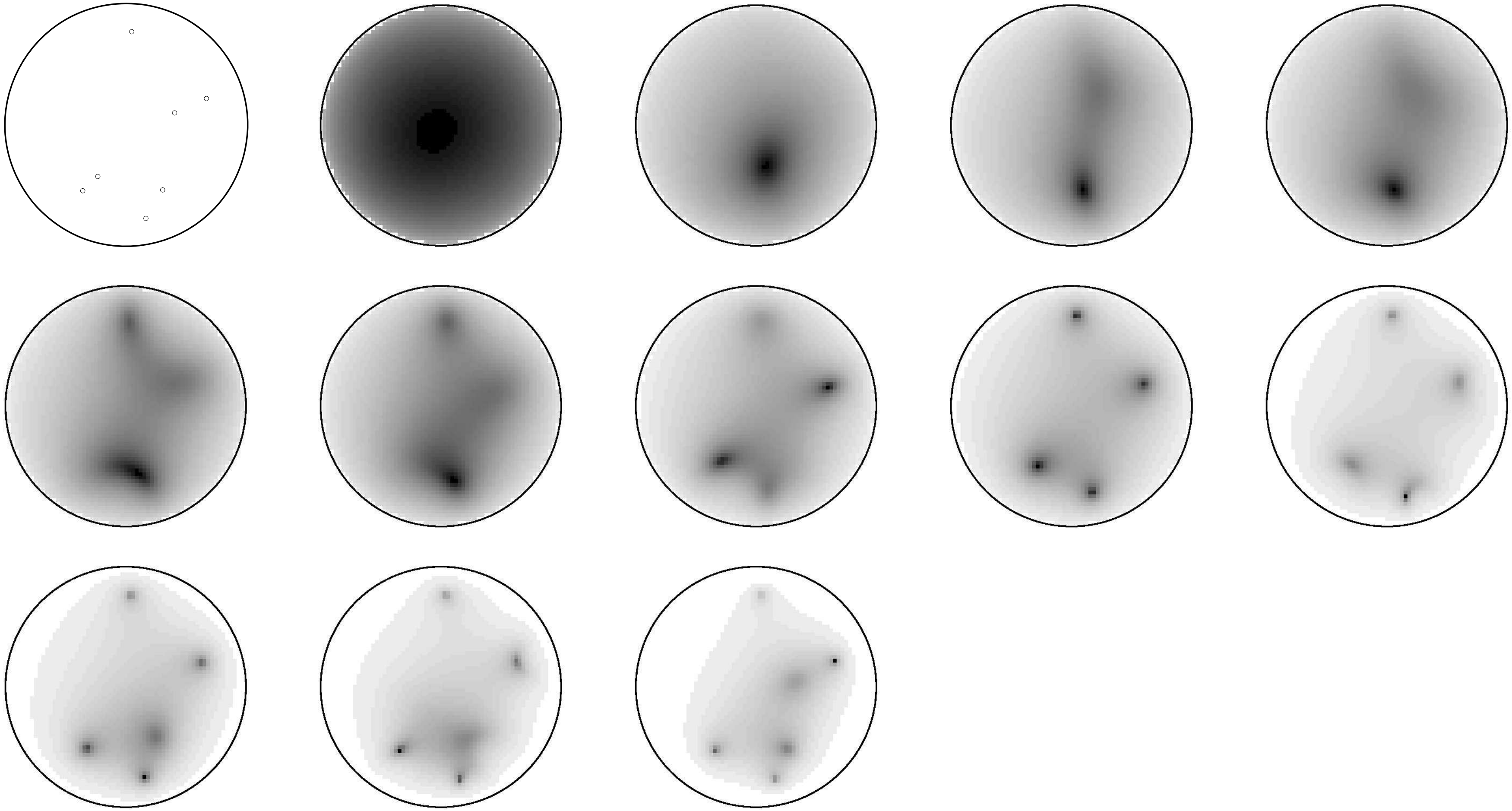

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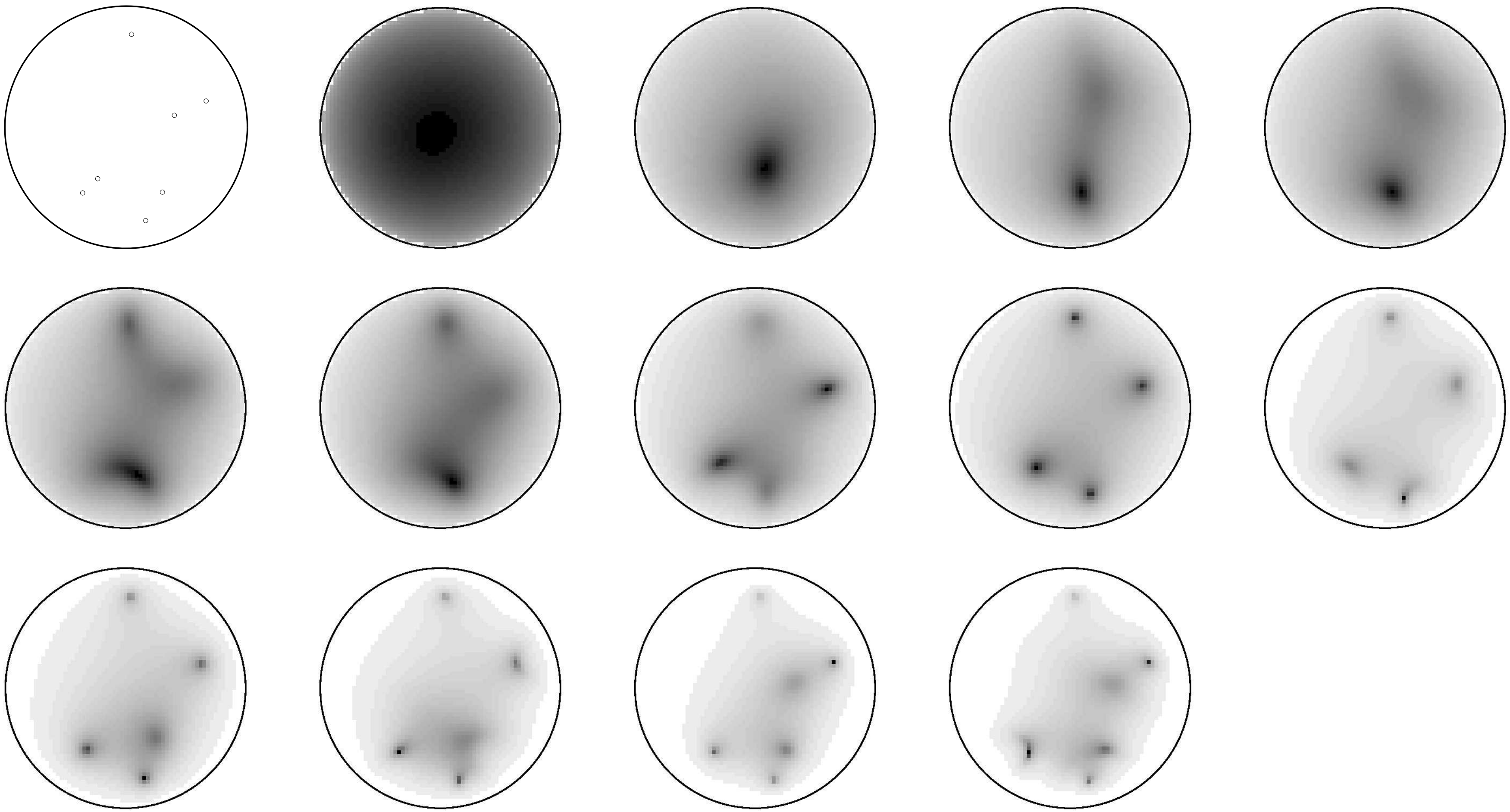

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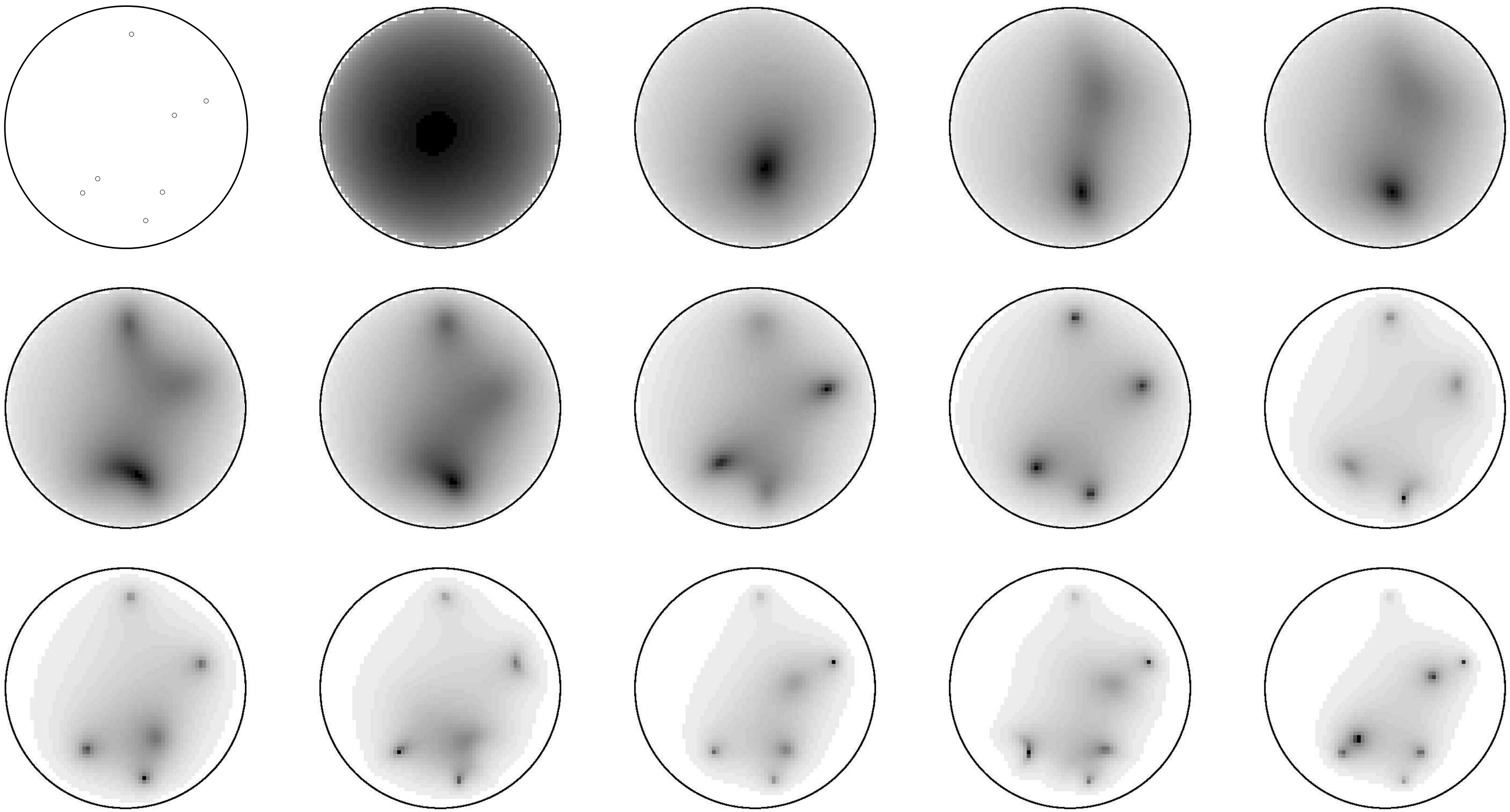

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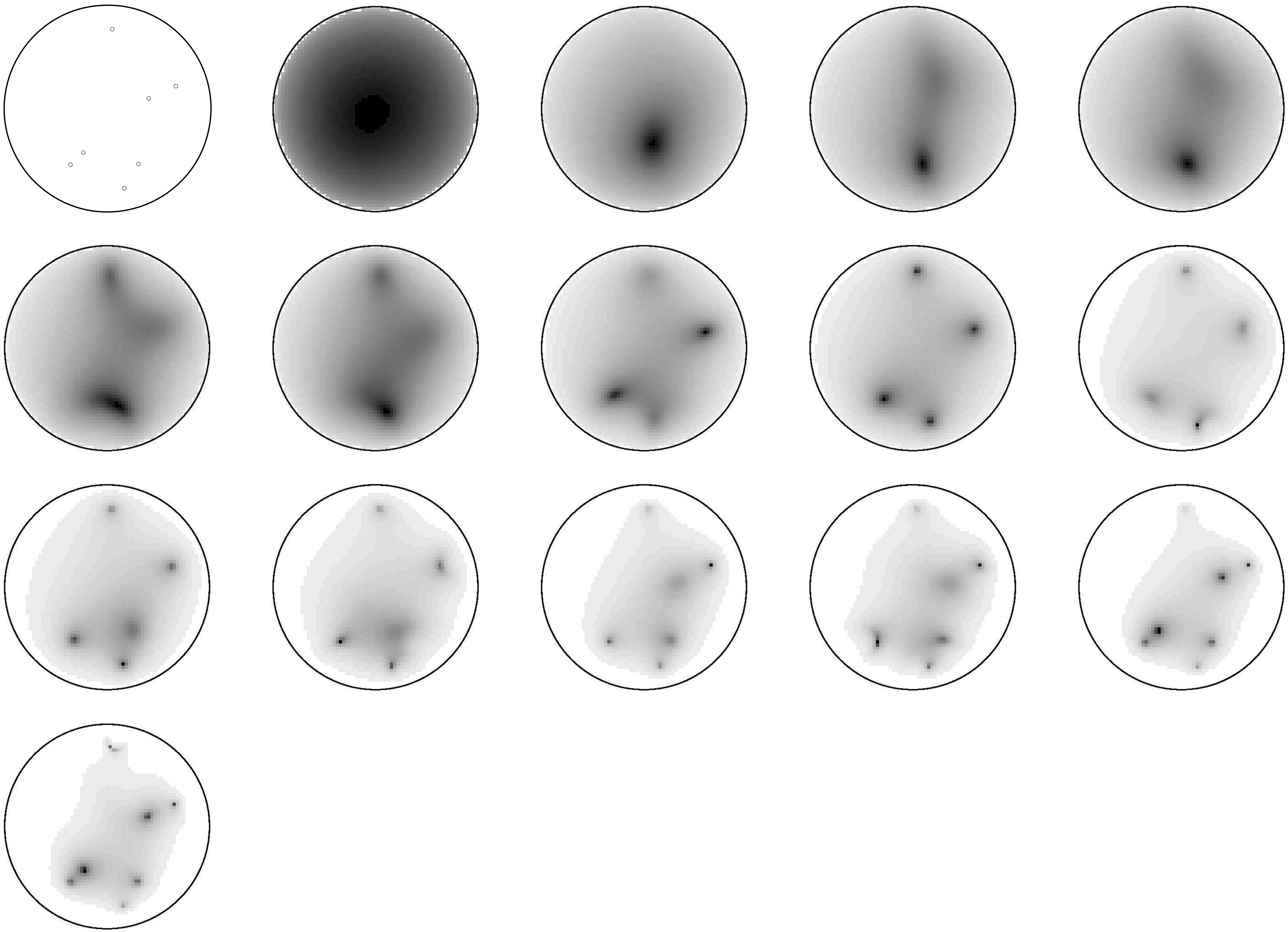

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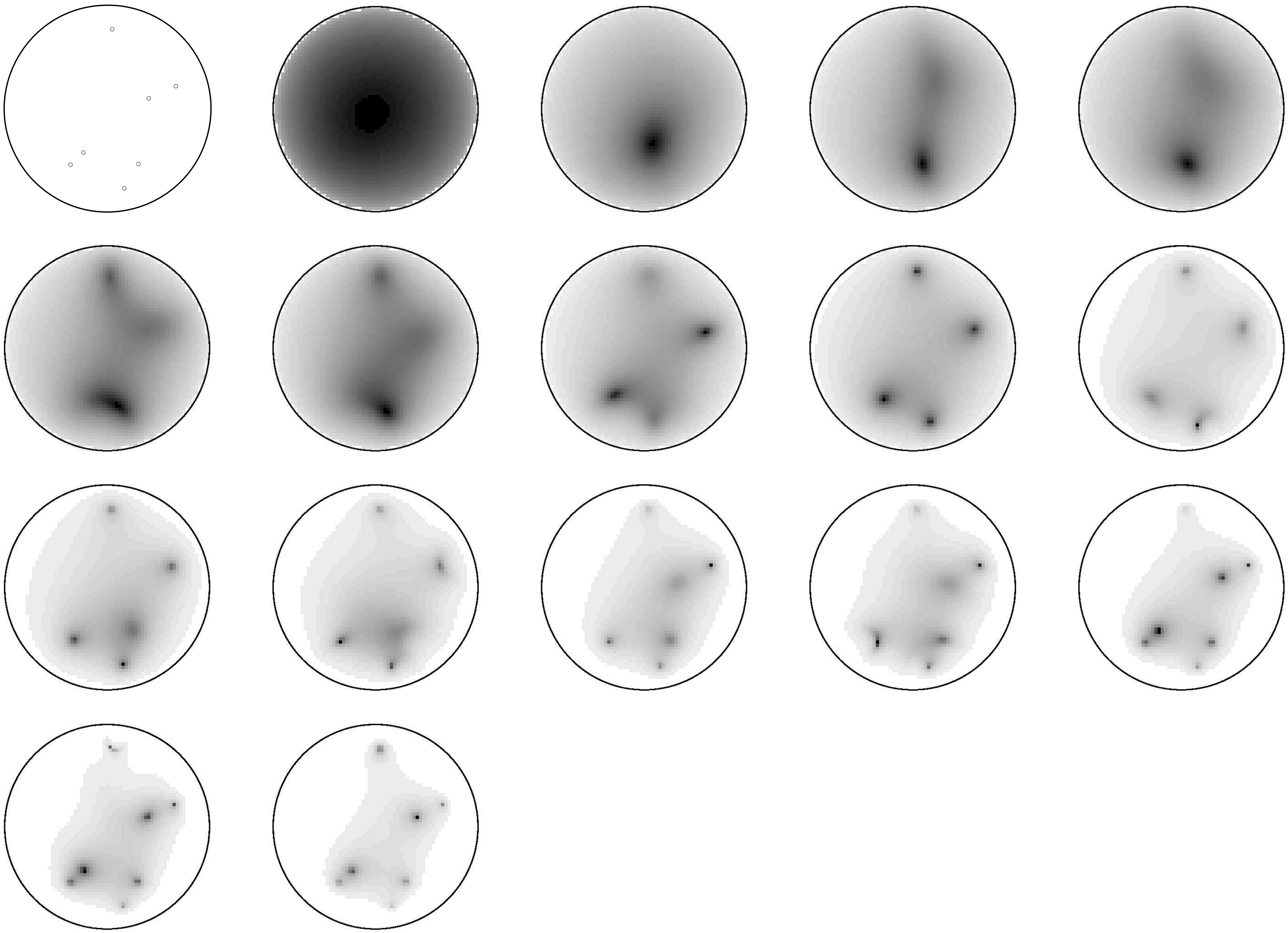

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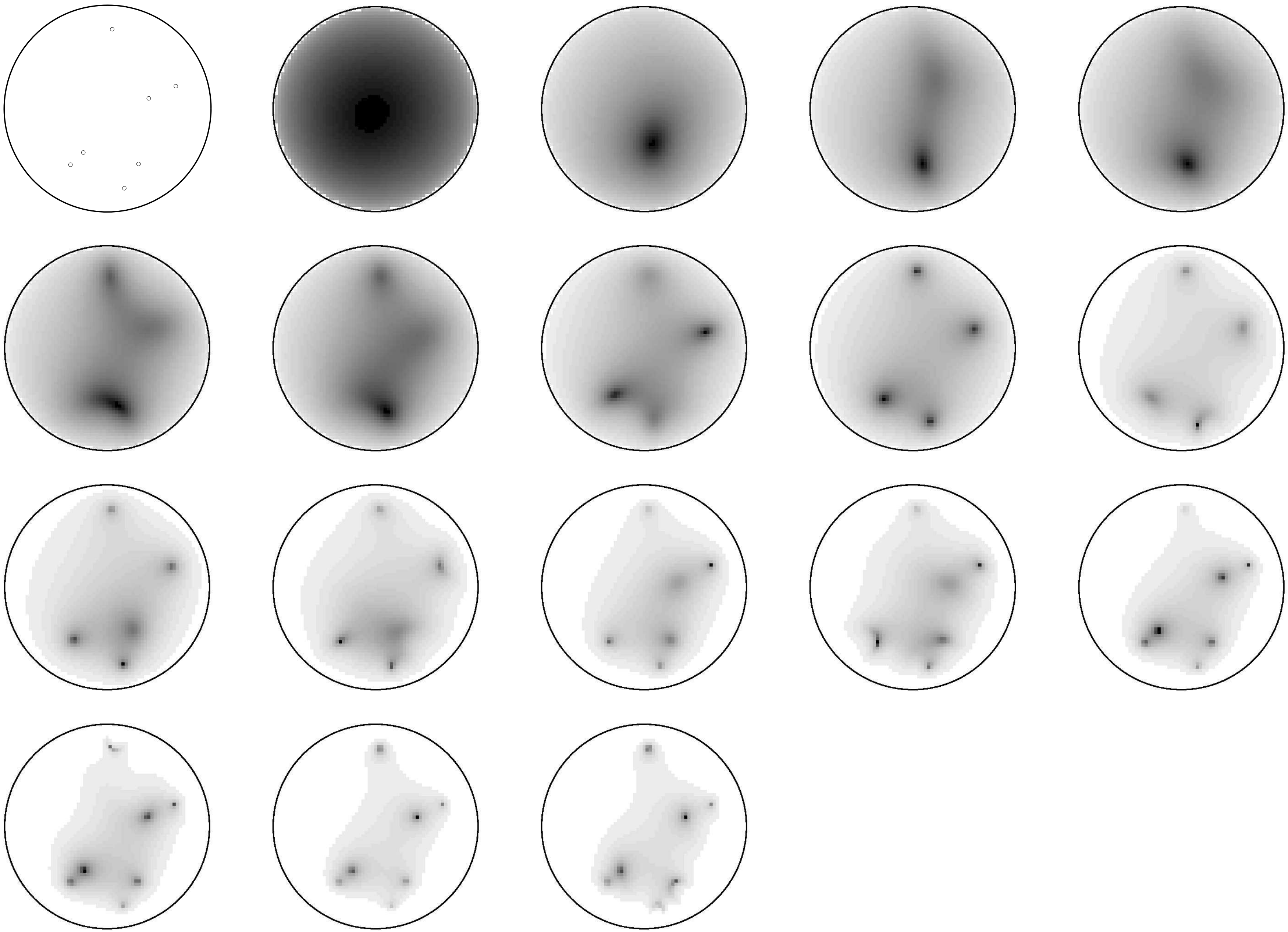

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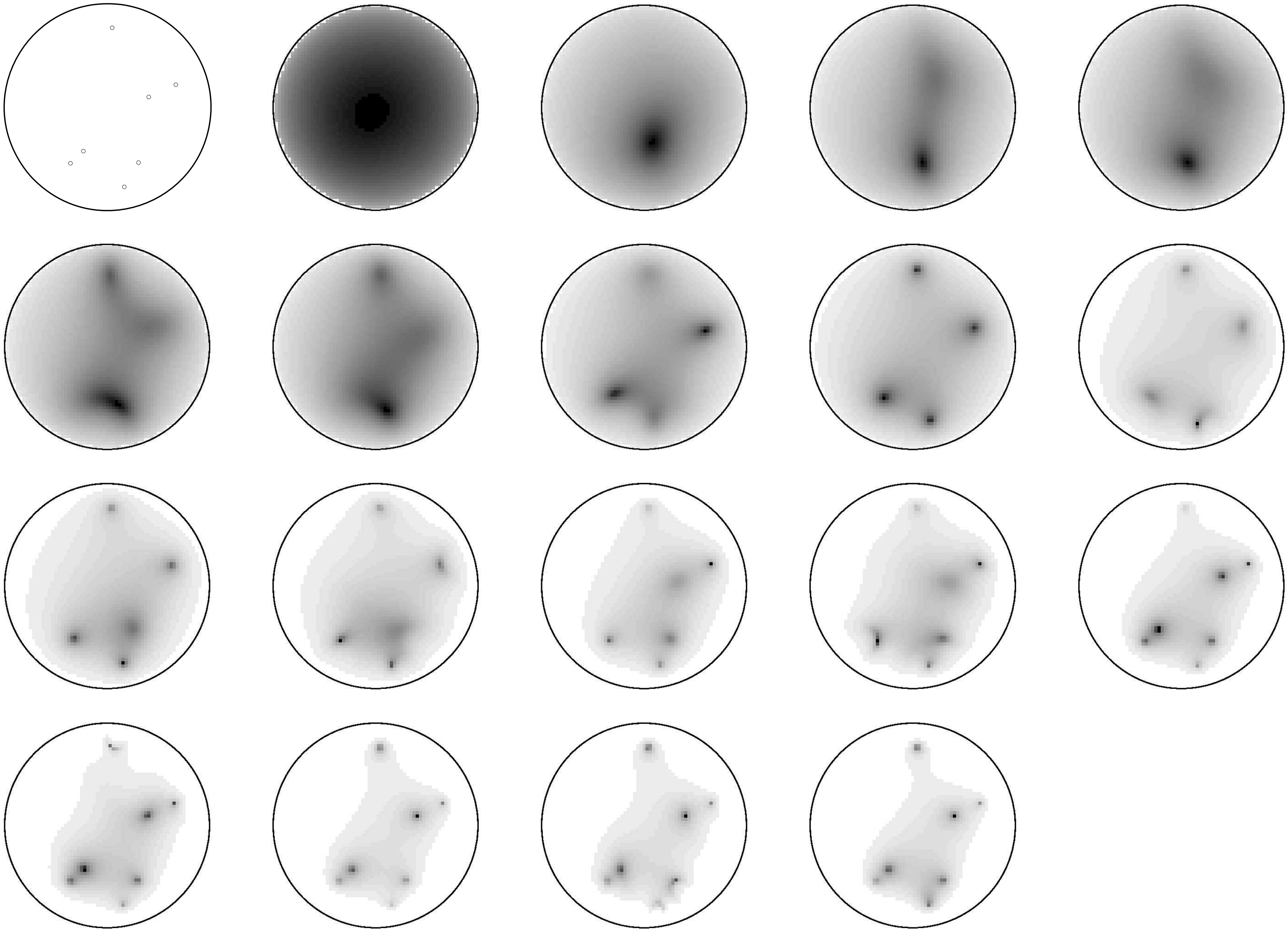

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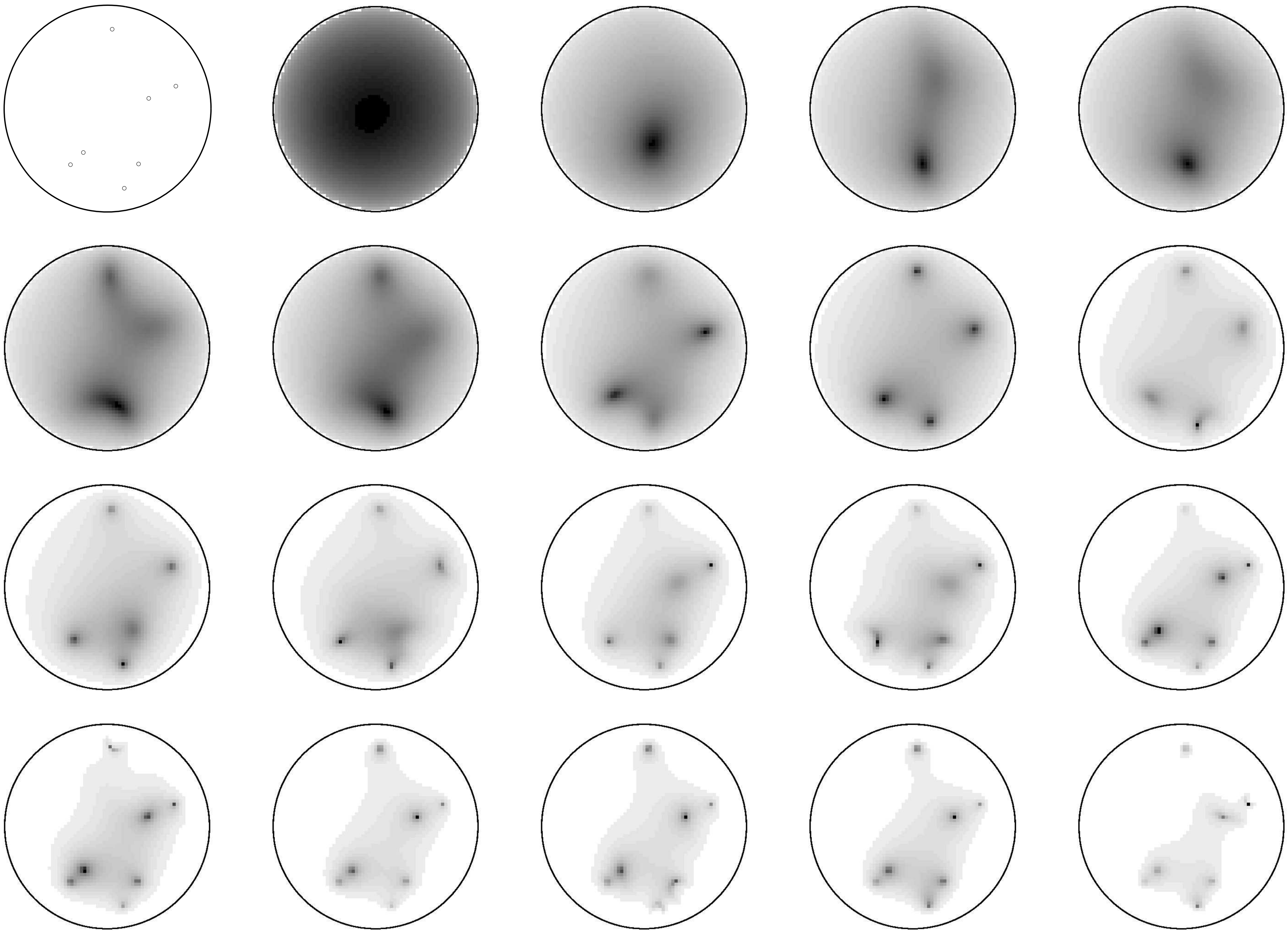

Volume estimation

Suppose γ_0 and γ_1 are constants.

$$u_\epsilon(y) - u_0(y) \quad = \quad |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}(x) \frac{\partial u_0}{\partial x_j} \frac{\partial N}{\partial x_i}(x, y) d\mu(x)$$

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Pick $\nabla u_0 = e_j$, $j = 1, \ldots, n$

$$\left| \frac{\text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \int_\Omega M_{jj} d\mu, \text{ and } \left| \frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0} \right| = |\omega_\epsilon| \text{trace}(\int_\Omega M d\mu).$$

Volume estimation

Using the bounds on M we obtain

$$\min\left(1, \frac{\gamma_1}{\gamma_0}\right)\left|\frac{\text{data}_j}{\gamma_1 - \gamma_0}\right| \leq |\omega_\epsilon| \leq \max\left(1, \frac{\gamma_1}{\gamma_0}\right)\left|\frac{\text{data}_j}{\gamma_1 - \gamma_0}\right|.$$

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One measurement bounds. Valid in general (Alessandrini, Rosset, Seo).

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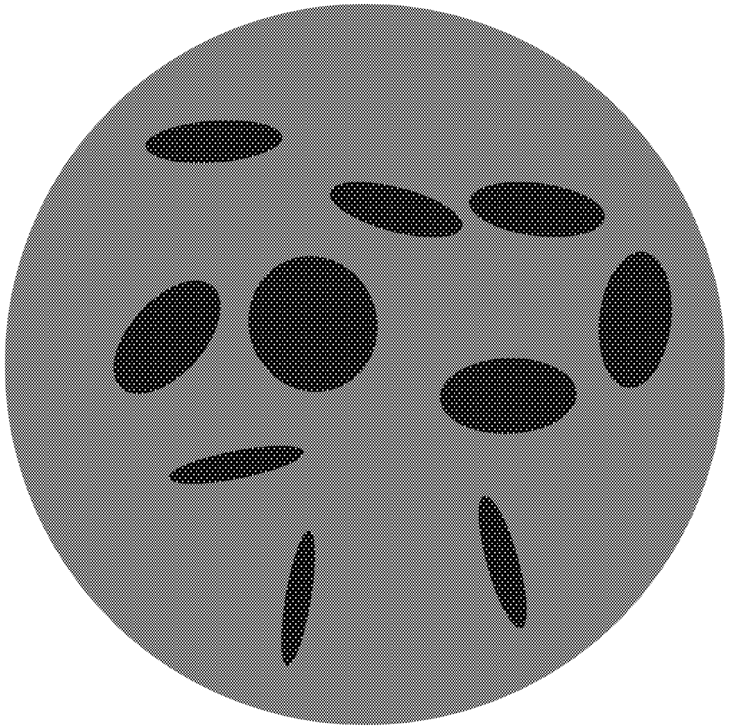
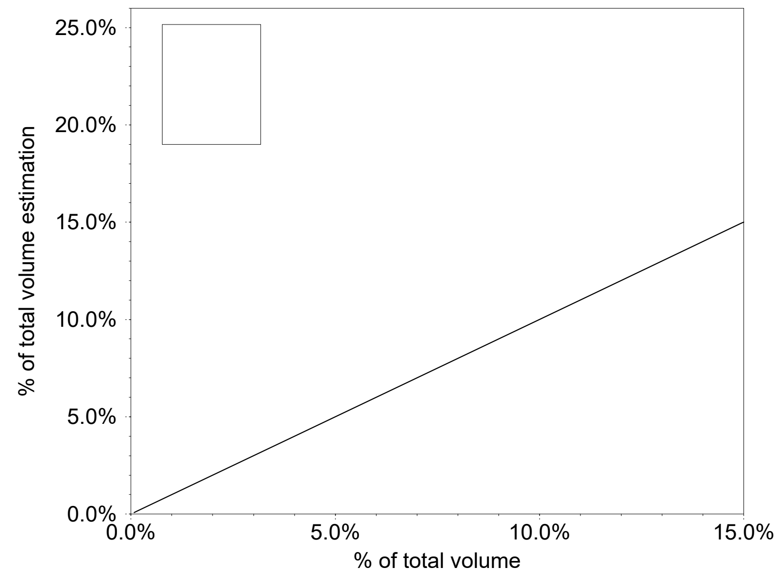
$$\frac{h\left(1, \ldots, 1, \frac{\gamma_1}{\gamma_0}\right)}{n} \left|\frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0}\right| \leq |\omega_\epsilon| \leq \frac{a\left(1, \ldots, 1, \frac{\gamma_1}{\gamma_0}\right)}{n} \left|\frac{\sum_{j=1}^n \text{data}_j}{\gamma_1 - \gamma_0}\right|$$

n measurement bounds. Only asymptotic bounds.

Random Ellipses

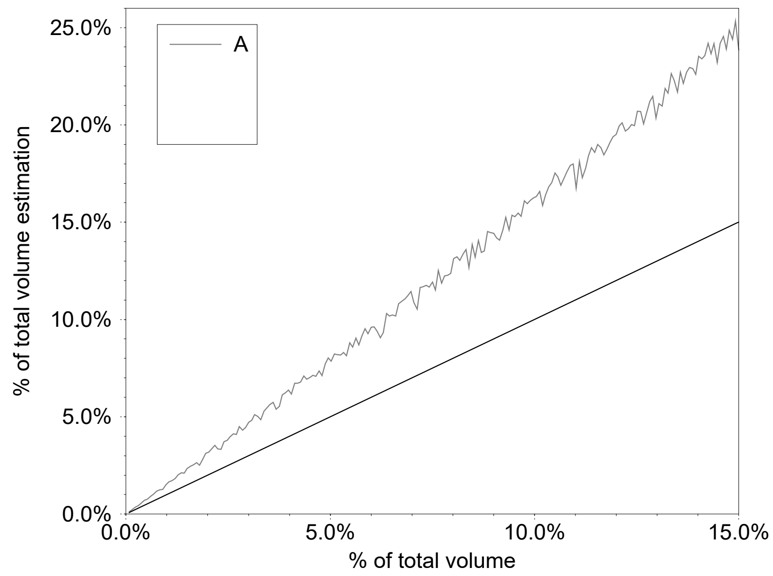


Volume estimation, with $\gamma_0 > \gamma_1$



Proportion of total volume occupied by the inhomogeneities.

Volume estimation, with $\gamma_0 > \gamma_1$



Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

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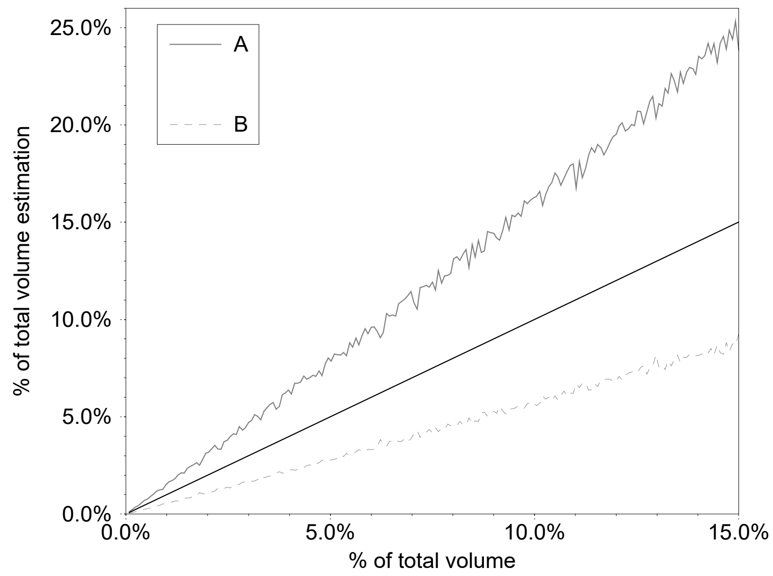
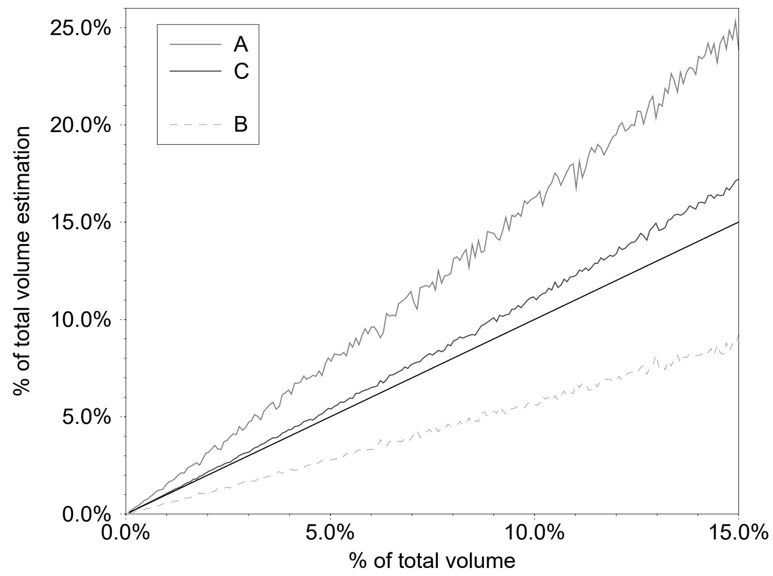


Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

Volume estimation, with $\gamma_0 > \gamma_1$



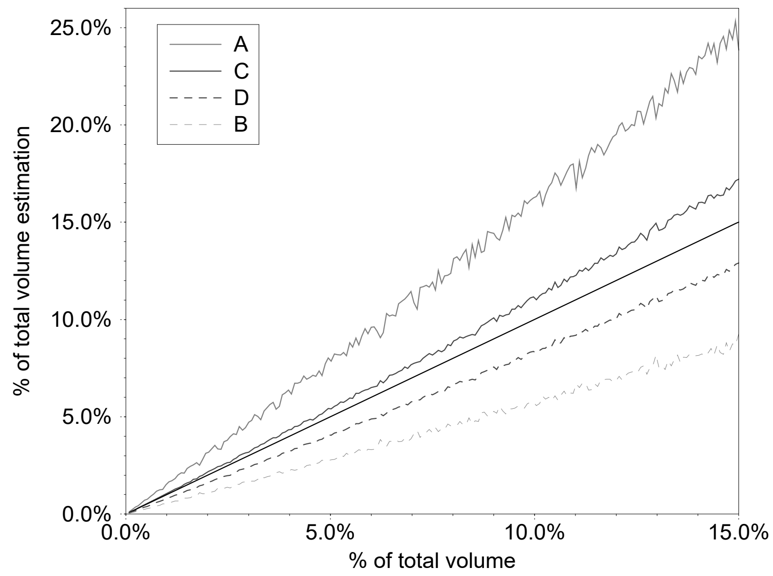
Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

C: the upper estimate obtained from two measurement bounds.

Volume estimation, with $\gamma_0 > \gamma_1$



Proportion of total volume occupied by the inhomogeneities.

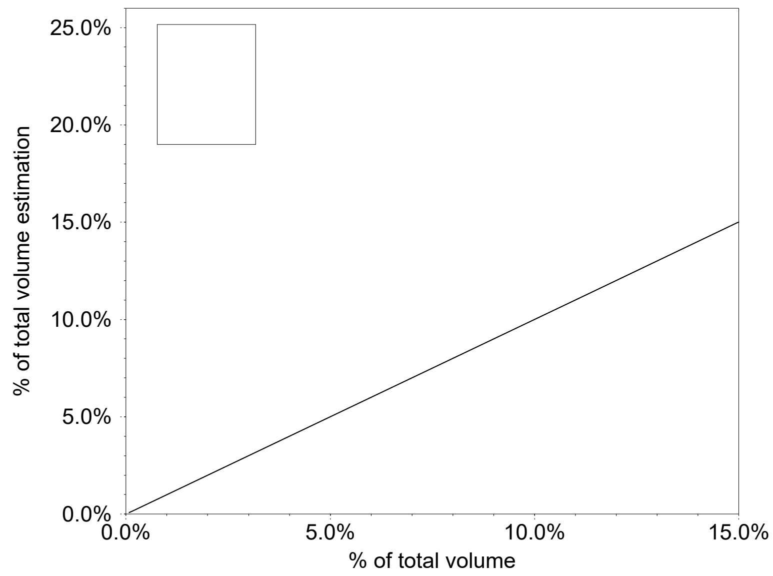
A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

C: the upper estimate obtained from two measurement bounds.

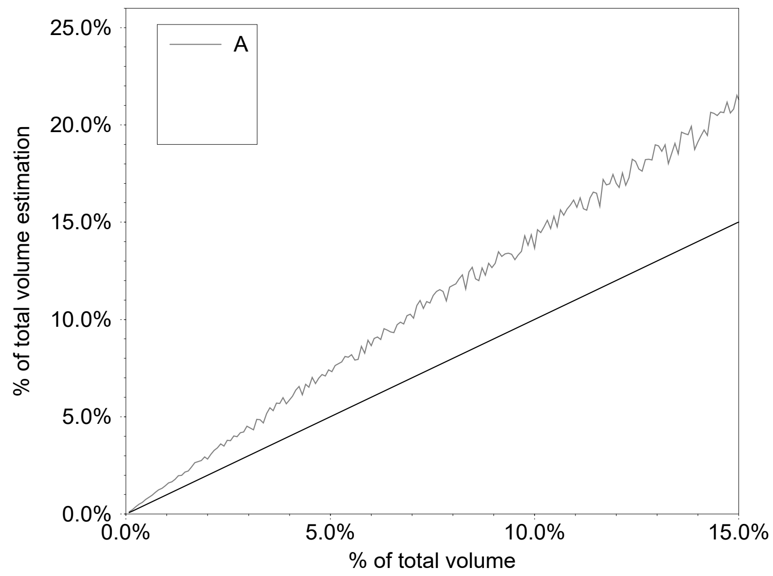
D: the lower estimate obtained from two measurement bounds.

Volume estimation, with $\gamma_1 > \gamma_0$



Proportion of total volume occupied by the inhomogeneities.

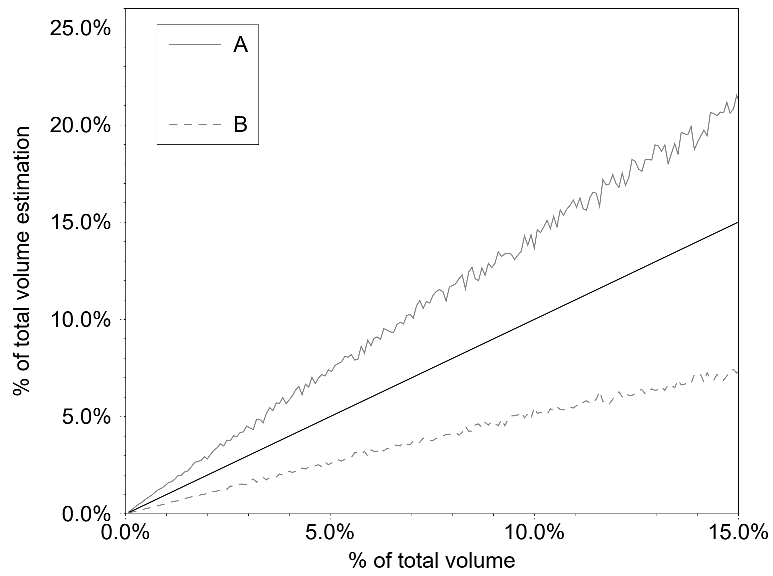
Volume estimation, with $\gamma_1 > \gamma_0$



Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

Volume estimation, with $\gamma_1 > \gamma_0$

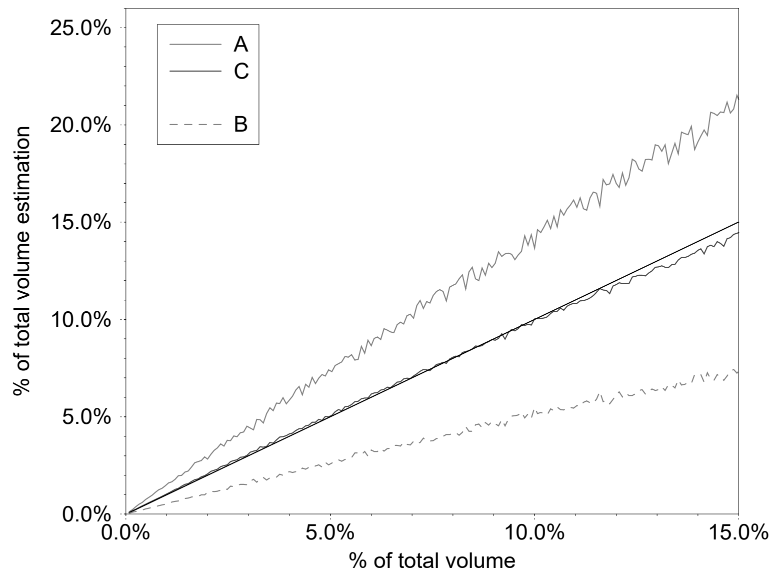


Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

Volume estimation, with $\gamma_1 > \gamma_0$



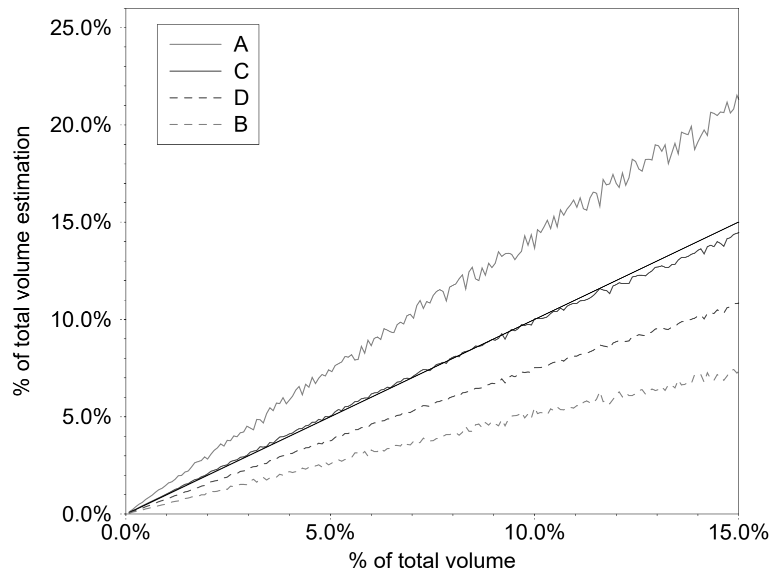
Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

C: the upper estimate obtained from two measurement bounds.

Volume estimation, with $\gamma_1 > \gamma_0$



Proportion of total volume occupied by the inhomogeneities.

A: the upper estimate obtained from one measurement bounds.

B: the lower estimate obtained from one measurement bounds.

C: the upper estimate obtained from two measurement bounds.

D: the lower estimate obtained from two measurement bounds.