Introduction to Differential Geometry

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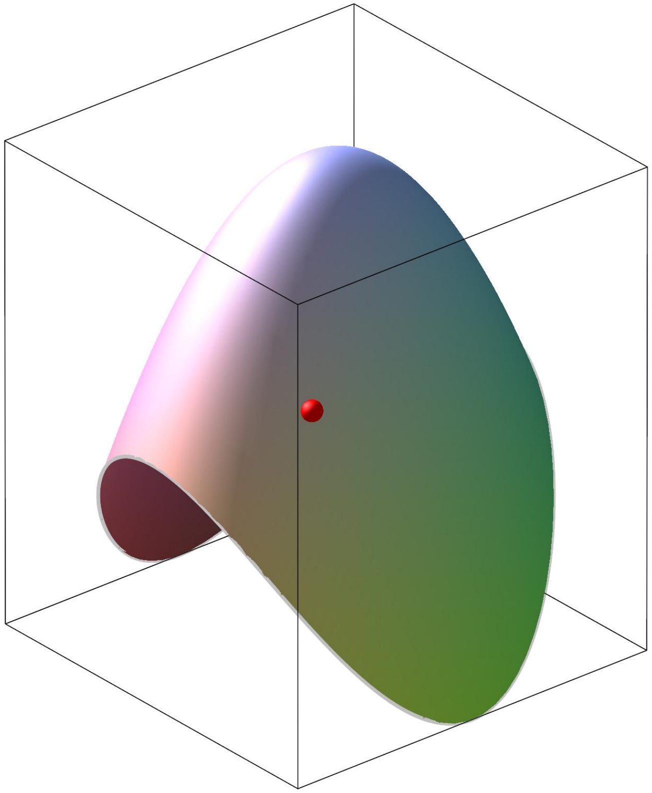
Overview

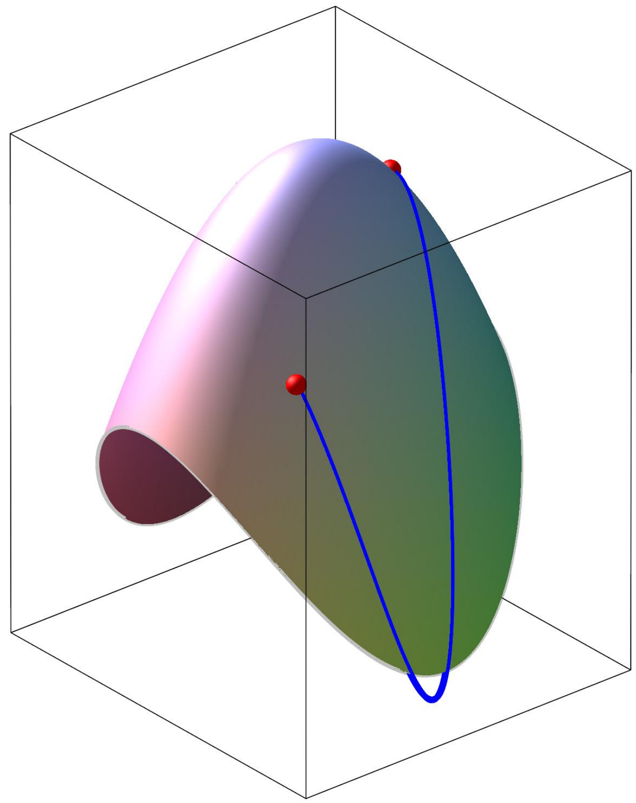
- Differential geometric aspects of surfaces in 3D (no formulas)
- Riemannian metrics, connections, and geodesics
- Conductive Riemannian manifolds and some associated PDEs
- Curvature issues

Formula-free aspects of surfaces

Formula-free aspects of surfaces

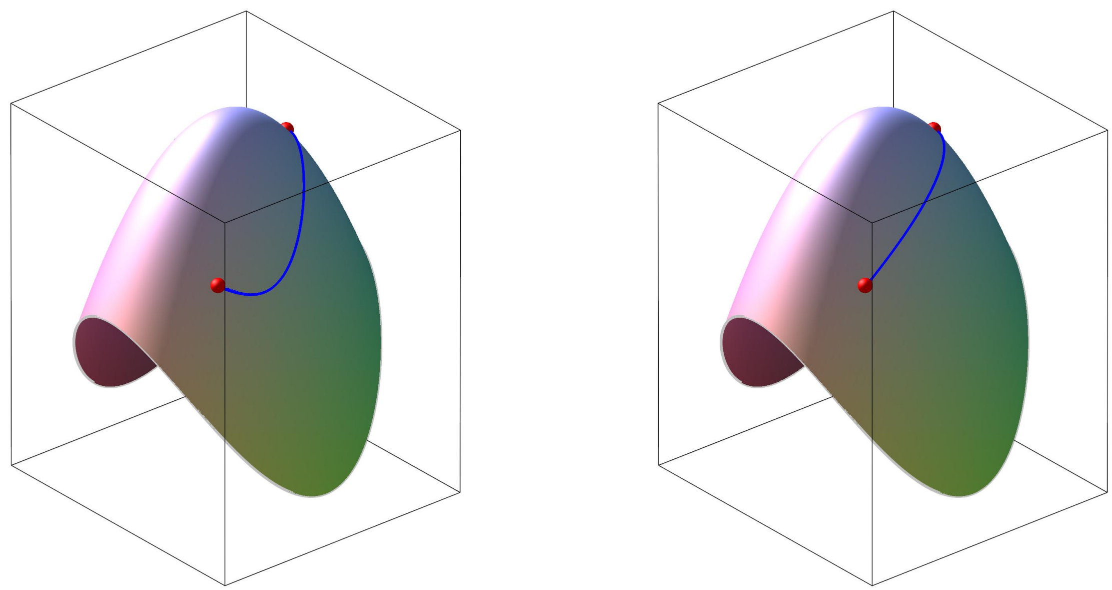


Formula-free aspects of surfaces



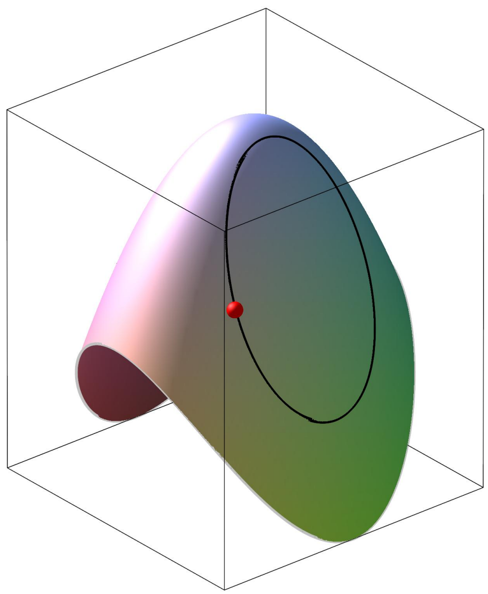
Formula-free aspects of surfaces

Theorem 1 (Hopf and Rinow, 1931)**.** *On a* comple-te *surface there is a unique geodesic (shortest curve) between any pair of (sufficiently close) points.*



Formula-free aspects of surfaces

Theorem 2 (Pestov and Uhlmann, 2005)**.** *In a* simple domain Ω *on a surface the boundary distances* $\mathrm{dist}_{\partial\Omega\times\partial\Omega}$ *determine* all *the distances* $\mathrm{dist}_{\Omega\times\Omega}$ *in the domain.*



Formula-free aspects of surfaces

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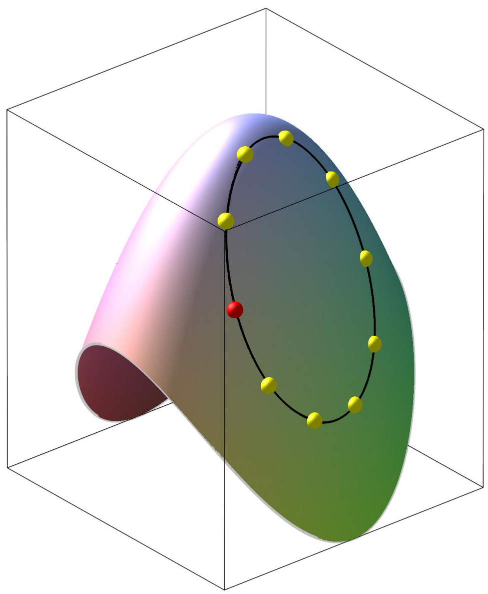

Formula-free aspects of surfaces

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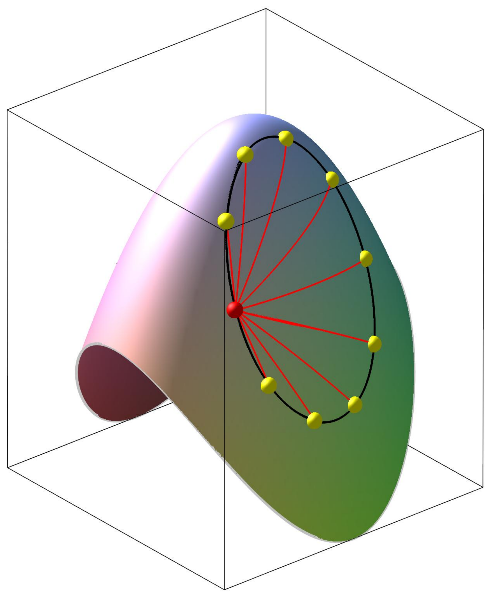

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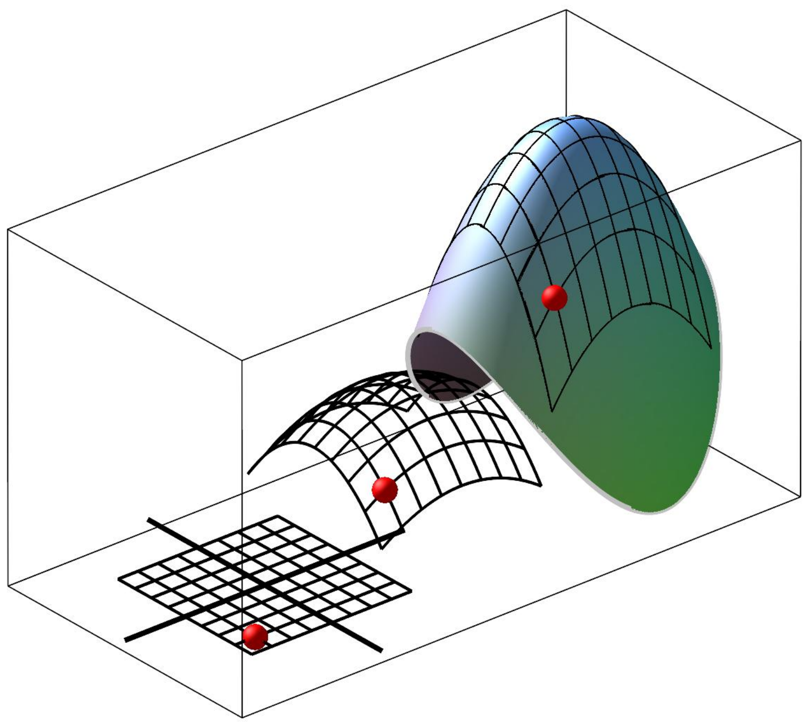
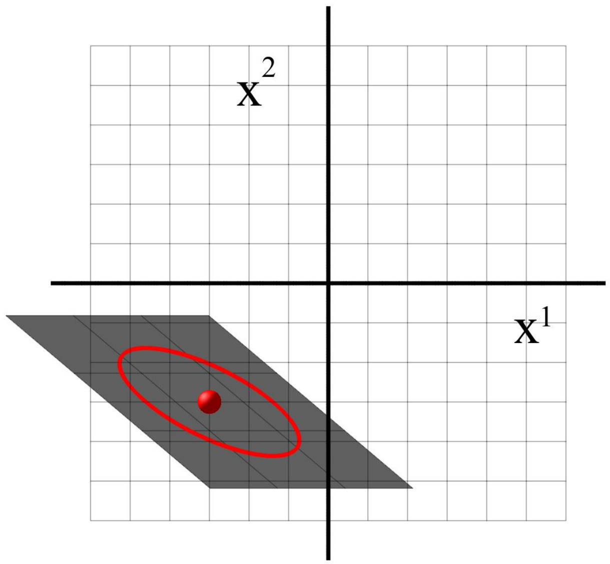
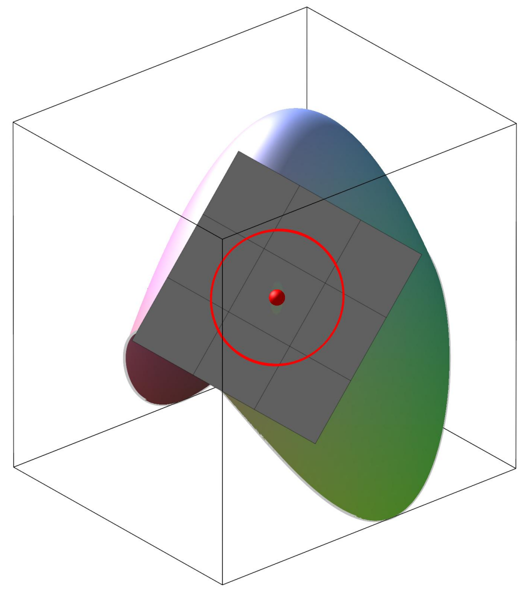
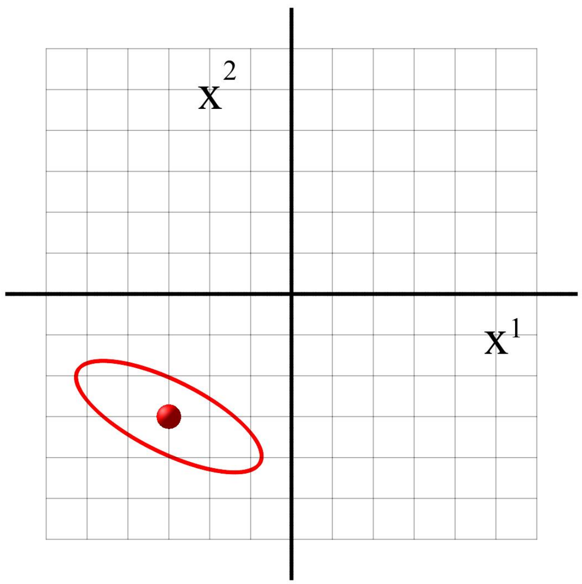


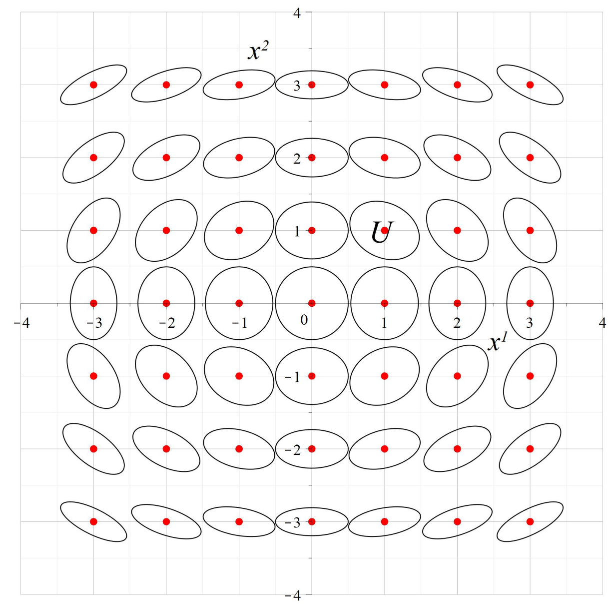
Pull-back of unit vectors





Indicatrix representation of the metric





Indicatrix representation of the metric

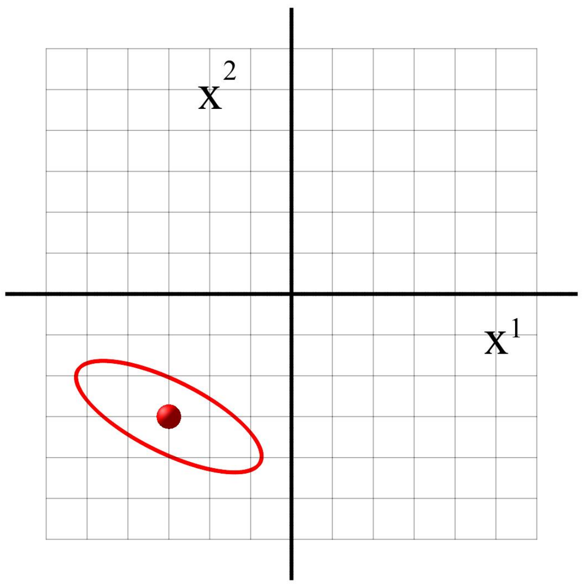
Principle. The indicatrix field in the coordinate domain $\mathcal{U}$ is equivalent to a *metric matrix field* in $\mathcal{U}$: At each point there is a unique quadratic form g with matrix

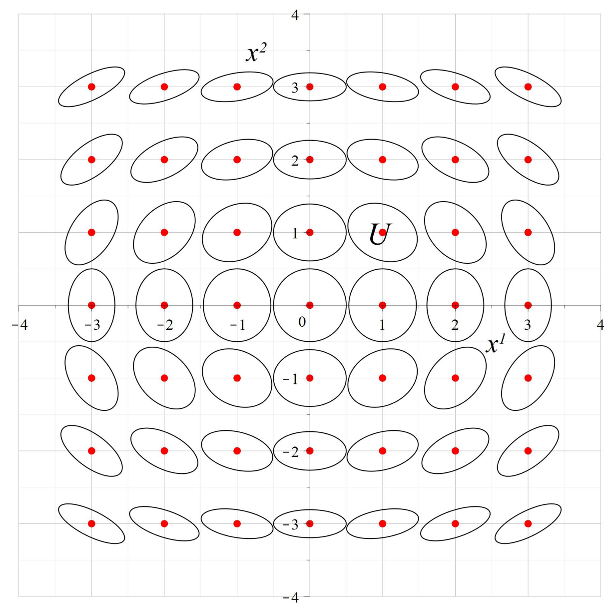
$$g_{i\,j} = g(e_i, e_j) \quad ,$$

so that the indicatrix at p is:

$$\mathcal{I}_p = \{V = \sum_k v^k \cdot e_k \in T_p\mathcal{U} \,|\, g(V,V) = \sum_{i\,j} g_{i\,j} \cdot v^i \cdot v^j = 1\} \quad .$$

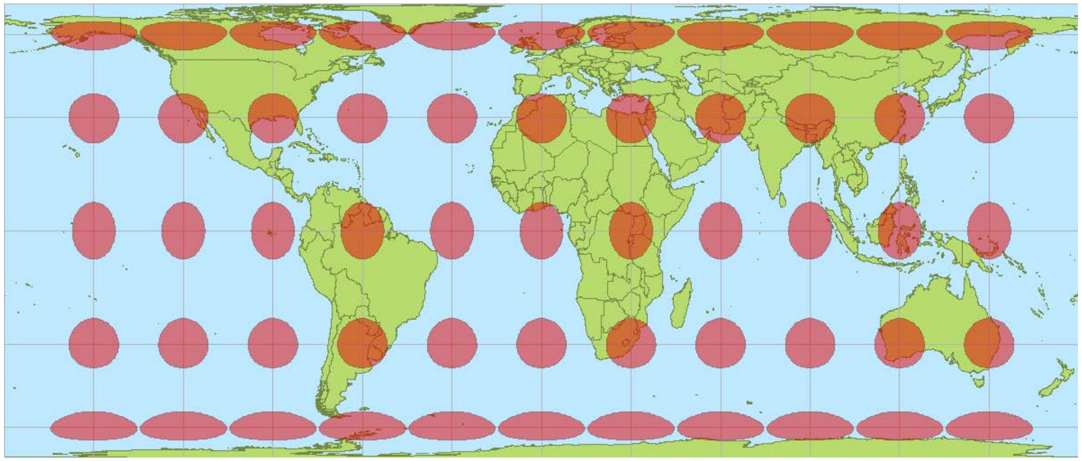
Indicatrix representation of the metric





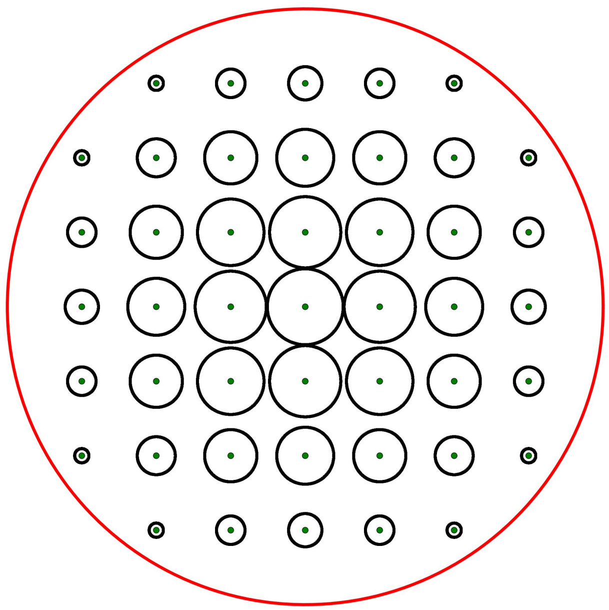
Indicatrix representation of the metric

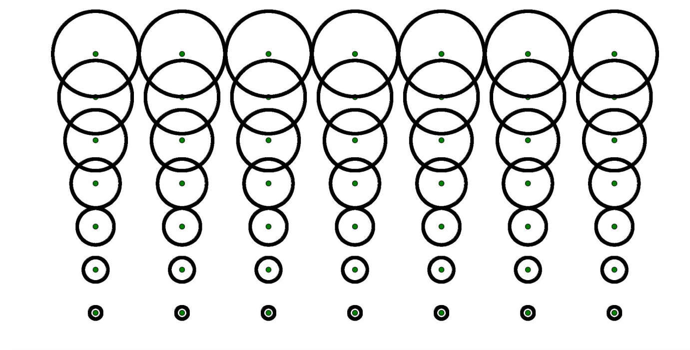
With Riemann we may consider metrics induced from well-known surfaces:



Indicatrix representation of the metric

– *as well as* metrics that are *not* inherited from surfaces in Euclidean $\mathbb{R}^3$, e.g. the Poincaré disk and half-plane:





Next: Work in $(\mathcal{U}^n, g)$

Principle. We can (and will) express everything in $(\mathcal{U}^n, g)$, $\mathcal{U} \subset \mathbb{R}^n$, in terms of the metric matrix function $g_p(e_i, e_j) = g_{i\,j}(x^1, x^2)$.

NB: In (the special) case of a parametrized surface $r(x^1, x^2)$, the matrix $g_{i\,j}(x^1, x^2)$ for the inherited pull-back metric is

$$g_{i\,j}(x^1, x^2) = J^*(x^1, x^2) \cdot J(x^1, x^2) \quad ,$$

where J is the Jacobian of the vector function r, i.e.

$$J(x^1, x^2) = \left[\frac{\partial r}{\partial x^1}\ \frac{\partial r}{\partial x^2}\right] \quad ,$$

and J^* denotes the transpose of J.

Riemannian key concepts

Work in $(\mathcal{U}^n, g)$

Observation. The usual derivative of a vector field $V(t) = V_{\gamma(t)}$ along a curve γ in $\mathbb{R}^n$ does NOT work:

$$\frac{d}{dt}(V(t))_{|t_0} = \lim_{t \to t_0} \left(\frac{V(t) - V(t_0)}{t - t_0} \right) \quad .$$

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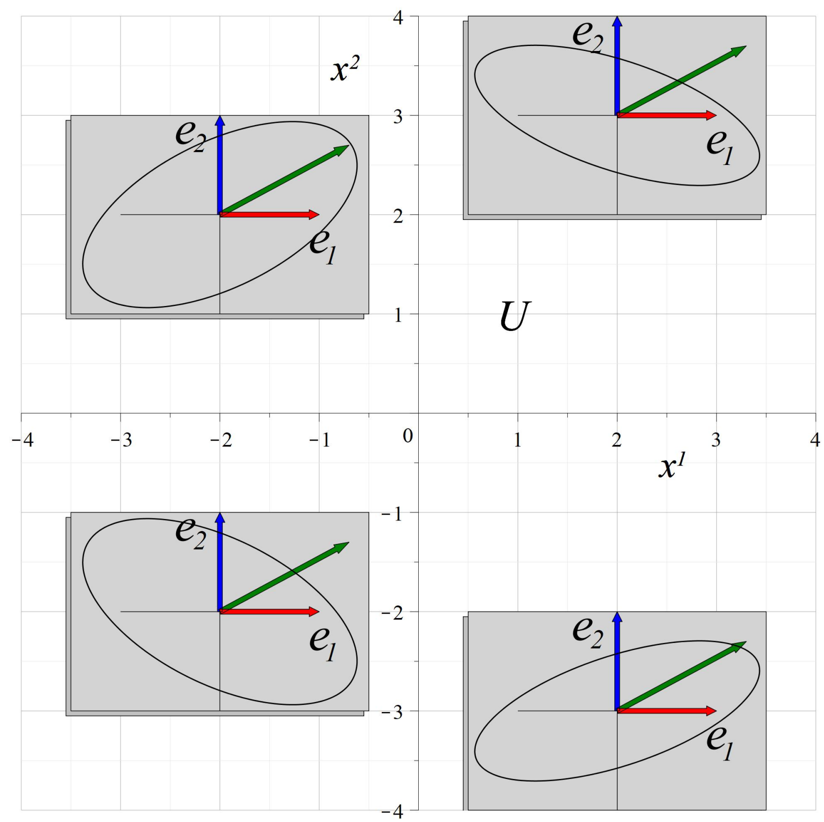
Because the two vectors $V(t)$ and $V(t_0)$ belong to different tangent spaces, $T_{\gamma(t)}\mathcal{U}$ and $T_{\gamma(t_0)}\mathcal{U}$, respectively, so they cannot be compared *directly*.

Work in $(\mathcal{U}^n, g)$



Work in $(\mathcal{U}^n, g, \nabla)$

We need a replacement, i.e. a modified *natural derivative* which is metric compatible, invariant, linear, Leibnizian:

$$\nabla_{\gamma'(t)} V$$

And, more generally, for any vector field X:

$$\nabla_X V$$

In $(\mathcal{U}^n, g)$ the replacement-solution is – in terms of standard base vector fields $\{e_1, \cdots, e_n\}$ – ***the Levi-Civita connection*** ∇:

$$\nabla_{e_i} e_j = \sum_k \Gamma^k_{i\,j} \cdot e_k \quad .$$

Work in $(\mathcal{U}^n, g, \nabla)$

The Christoffel symbol functions are defined by the metric matrix functions as follows, with $\left[g^{\ell\, k}\right] = \left[g_{i\, j}\right]^{-1}$:

$$\Gamma^{k}_{i\, j}(x^1, x^2) = \frac{1}{2} \cdot \sum_{\ell=1}^{\ell=n} \left(\frac{\partial}{\partial x^i} g_{j\, \ell} + \frac{\partial}{\partial x^j} g_{\ell\, i} - \frac{\partial}{\partial x^\ell} g_{i\, j} \right) \cdot g^{\ell\, k} \quad .$$

Observation. If g is Euclidean, i.e. $\left[g_{i\, j}\right]$ is a constant matrix function, then

$$\Gamma^{k}_{i\, j}(x^1, x^2) = 0 \quad \text{for all } (x^1, x^2) \in \mathcal{U}$$

Work in $(\mathcal{U}^n, g, \nabla)$

With these ingredients, i.e. in $(\mathcal{U}, g, \nabla)$, we then get, for $V = \sum_j v^j \cdot e_j$:

$$\nabla_{\gamma'(t)} V = \sum_k \left(\frac{dv^k}{dt} + \sum_{i\,j} v^j(t) \cdot (\gamma^{\,i})'(t) \cdot \Gamma^k_{i\,j}(\gamma(t)) \right) \cdot e_k$$

and for $X = \sum_j x^j \cdot e_j$

$$\nabla_X V = \sum_{i\,j\,k} \left(x^i \cdot v^j \cdot \Gamma^k_{i\,j} + X(v^k) \right) \cdot e_k$$

Work in $(\mathcal{U}^n, g, \nabla)$

Observation. If g is Euclidean, then we get the well-known 'usual' expressions:

$$\nabla_{\gamma'(t)} V = \frac{d}{dt} V(\gamma(t)) = \sum_k \frac{dv^k}{dt} \cdot e_k$$

and

$$\nabla_X V = \sum_k X(v^k) \cdot e_k$$

Work in $(\mathcal{U}^n, g, \nabla)$

From here we can now define the first key operators:

$$\mathsf{grad}_g(f) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} g^{k\,\ell} \cdot \frac{\partial f}{\partial x^\ell} \cdot e_k$$

$$\begin{aligned}
\mathsf{div}_g(V) &= \sum_k \sum_\ell g(\nabla_{e_\ell} V, e_k) \cdot g^{k\,\ell} \\
&= \sum_i \frac{\partial}{\partial x^i} v^i + \sum_i \sum_j \Gamma^i_{i\,j} \cdot v^j \\
\Delta_g(f) &= \mathsf{div}_g(\mathsf{grad}_g(f))
\end{aligned}$$

Work in $(\mathcal{U}^n, g, \nabla)$

En passant: If $g = g_E$ is Euclidean, then the expressions reduce to well-known identitites:

$$\text{grad}_{g_E}(f) = \sum_{k=1}^{n} \frac{\partial f}{\partial x^k} \cdot e_k$$

$$\text{div}_{g_E}(V) = \sum_i \frac{\partial}{\partial x^i} v^i$$

$$\Delta_{g_E}(f) = \sum_i \frac{\partial^2 f}{\partial (x^i)^2}$$

Work in $(\mathcal{U}^n, g, \nabla)$

Theorem 5. *Geodesics, i.e. the* distance realizing curves γ *in* $(\mathcal{U}^n, g, \nabla)$, *are precisely the auto-parallel unit speed curves – they satisfy the differential equation (system) obtained by first variation of arc-length:*

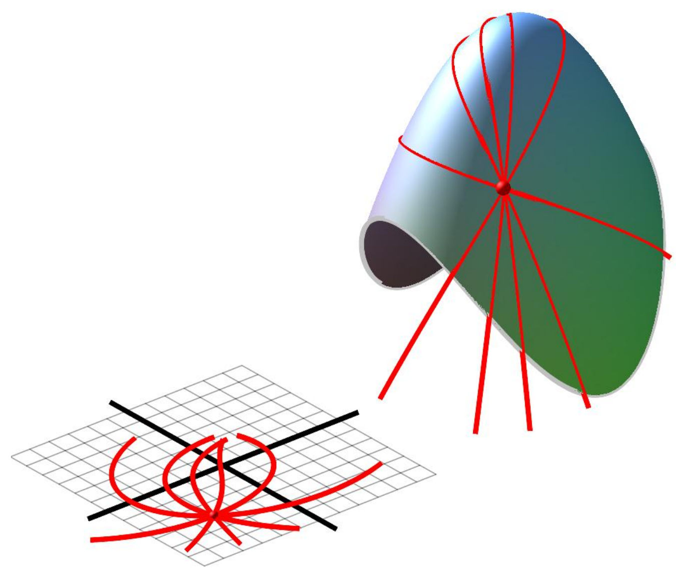
$$\nabla_{\gamma'}\gamma'(s) = 0 \quad .$$

Equivalently, in $\mathcal{U}^n$ *coordinates:*

$$\sum_k \left(\frac{d^2}{ds^2}\gamma^k(s) + \sum_{ij} (\gamma^i)'(s) \cdot (\gamma^j)'(s) \cdot \Gamma^k_{ij}(\gamma(s)) \right) \cdot e_k = 0$$

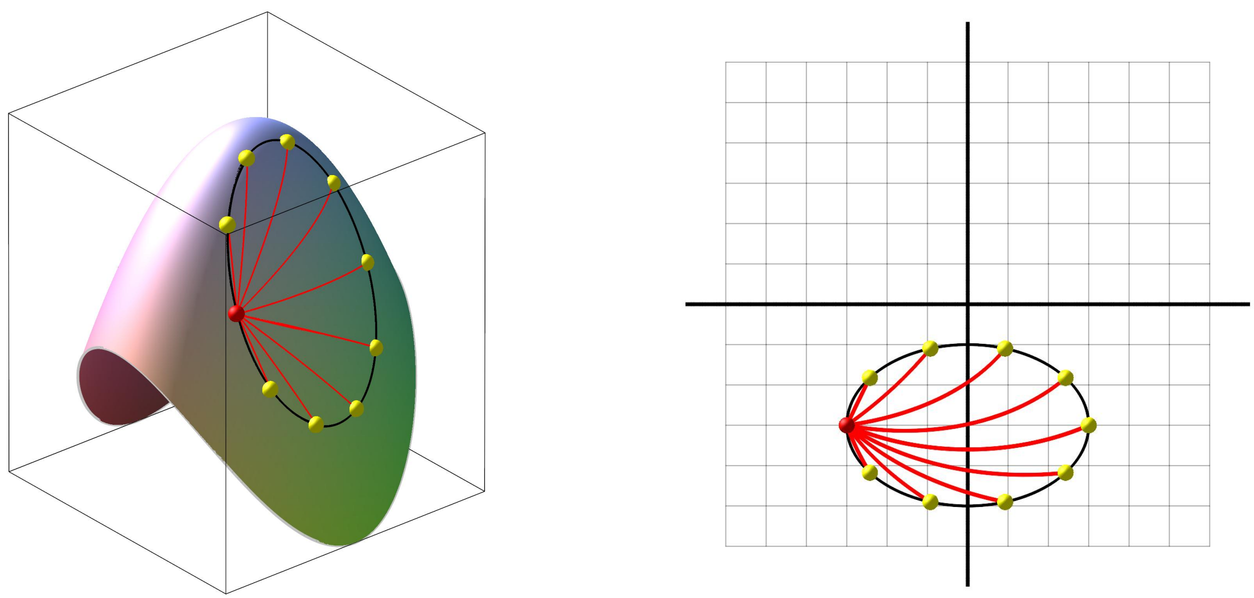
Geodesics

The geodesic spray from a point – on the surface and in the isometric copy $(\mathcal{U}^n, g, \nabla)$:



Geodesics

Flashback: The boundary rigidity setting on the surface and in $(\mathcal{U}^2, g)$:



Conductive Riemannian manifolds

Conductive Riemannian manifolds

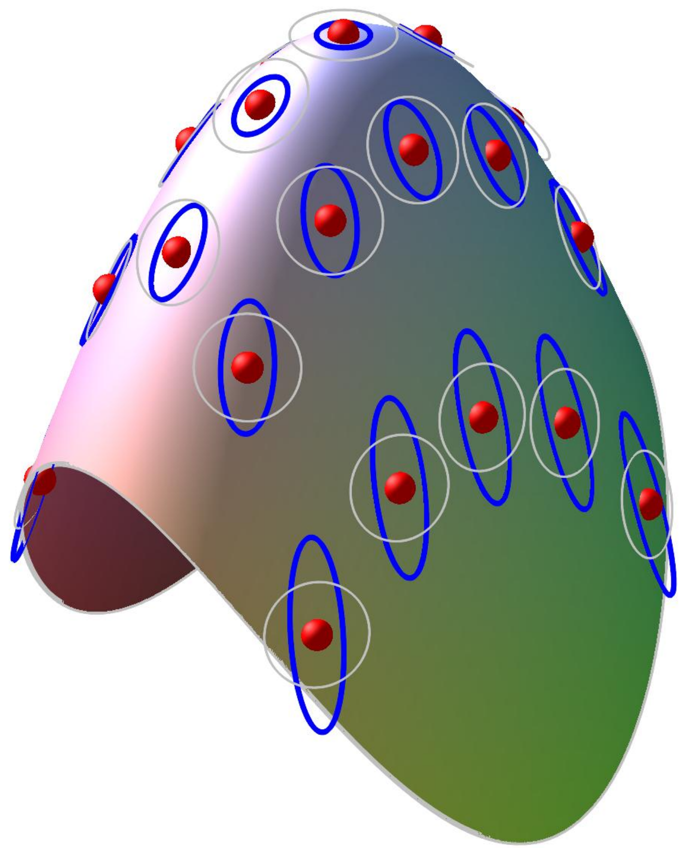
Definition. A conductivity $\mathcal{W}$ on a local Riemannian manifold $(\mathcal{U}, g, \nabla)$ is a (smooth) assignment of at linear map $\mathcal{W}$ to each tangent space $T_p\mathcal{U}$.

We assume that $\mathcal{W}$ is everywhere self-adjoint and positive definite with respect to the metric g:

$$g\,(\mathcal{W}(X), Y) = g\,(X, \mathcal{W}(Y))$$

$g\,(\mathcal{W}(X), X) \geq \xi^2 \cdot g(X, X)$ for some non-zero constant ξ .

Conductive Riemannian manifolds



Conductive Riemannian manifolds

In $\mathcal{U}^n$-coordinates:

$$\mathcal{W}(e_j) = \sum_k \mathcal{W}_j^k \cdot e_k$$

$$\mathcal{W}(U) = \mathcal{W}\left(\sum_j u^j \cdot e_j\right) = \sum_j u^j \cdot \mathcal{W}(e_j) = \sum_{j\,k} u^j \cdot \mathcal{W}_j^k \cdot e_k$$

Conductive Riemannian manifolds

Physical interpretation: Any given *potential function* f on $(\mathcal{U}, g)$ has a gradient vector field $\mathsf{grad}_g(f)$ which is turned into a *current vector field* I by $\mathcal{W}$:

$$\begin{aligned} I &= \mathcal{W}(\mathsf{grad}_g(f)) \\ &= \sum_{j\,k\,\ell} \mathcal{W}_j^k \cdot g^{j\,\ell} \cdot \frac{\partial f}{\partial x^\ell} \cdot e_k \end{aligned}$$

Conductive Riemannian manifolds

Current vector fields have *zero divergence*. We therefore define the $\mathcal{W}$-modified geometric Laplacian to express this fact for potential functions f:

Definition.

$$\Delta_g^{\mathcal{W}}(f) = \mathrm{div}_g\left(\mathcal{W}(\mathrm{grad}_g(f))\right) = 0 \quad .$$

Observation. The operator $\Delta_g^{\mathcal{W}}(f)$ is (still) linear and elliptic. It opens up for generalizations of many aspects of potential theory on Riemannian manifolds – and on weighted Riemannian manifolds.

The mean exit time

Definition. The $\mathcal{W}$-driven *mean exit time function* u from a compact domain Ω in a Riemannian manifold (M^n, g, ∇) is the unique solution to the $\mathcal{W}$-modified Poisson boundary value problem:

$$\Delta_g^{\mathcal{W}}(u) = -1 \quad , \quad u_{|_{\partial\Omega}} = 0 \quad .$$

Quest. For which conductivities is it possible to derive curvature dependent comparison theorems for such mean exit time functions? For isotropic conductivities this question is related to the recently much studied analysis of *weighted manifolds*.

The Calderón problem

Definition. The Riemannian Calderón problem is concerned with the Dirichlet problem for $\Delta_g^{\mathcal{W}}$ on a compact domain Ω in a Riemannian manifold (M^n, g, ∇):

$$\Delta_g^{\mathcal{W}}(u) = 0 \quad , \quad u_{|\partial\Omega} = f \quad .$$

Quest. Suppose the metric g is given. The problem is to reconstruct the conductivity $\mathcal{W}$ from knowledge of the Dirichlet-to-Neumann map (with $\nu =$ the unit inward pointing normal vector field at the boundary):

$$\Lambda_g^W \; : \; f \mapsto \partial_\nu u_{|\partial\Omega}$$

What is curvature?

Curvature

Definition. The curvature tensor $\mathcal{R}$ on a Riemannian manifold (M^n, g, ∇) is defined via g (and the induced connection ∇) on 4 vector fields:

$$\mathcal{R}(X,Y,V,U) = g\left(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V \, , \, U\right) \quad ,$$

where $[X, Y]$ is the Lie bracket (vector field) derivation on functions:

$$[X, Y]\,(f) = X(Y(f)) - Y(X(f)) \quad .$$

Curvature

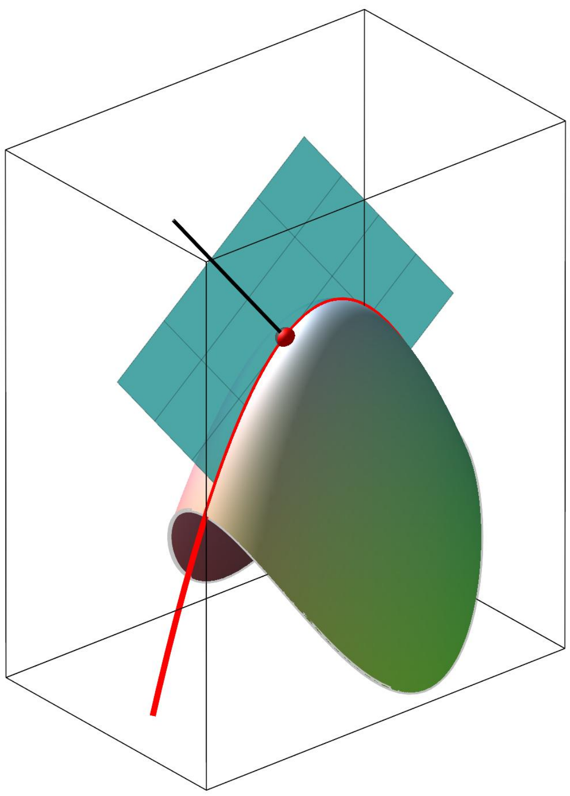
Definition. The sectional curvature is a function on the set of regular two-planes $\sigma = \text{span}(X, Y)$ in each tangent space $T_p\mathcal{U}$:

$$K(X,Y) = \frac{\mathcal{R}(X,Y,Y,X))}{\text{Area}^2(X,Y)} \quad ,$$

where $\text{Area}^2(X,Y)$ denotes the squared area of σ:

$$\text{Area}(X,Y) = g(X,X) \cdot g(Y,Y) - g^2(X,Y) \quad .$$

Euler cutting for curvature of a surface



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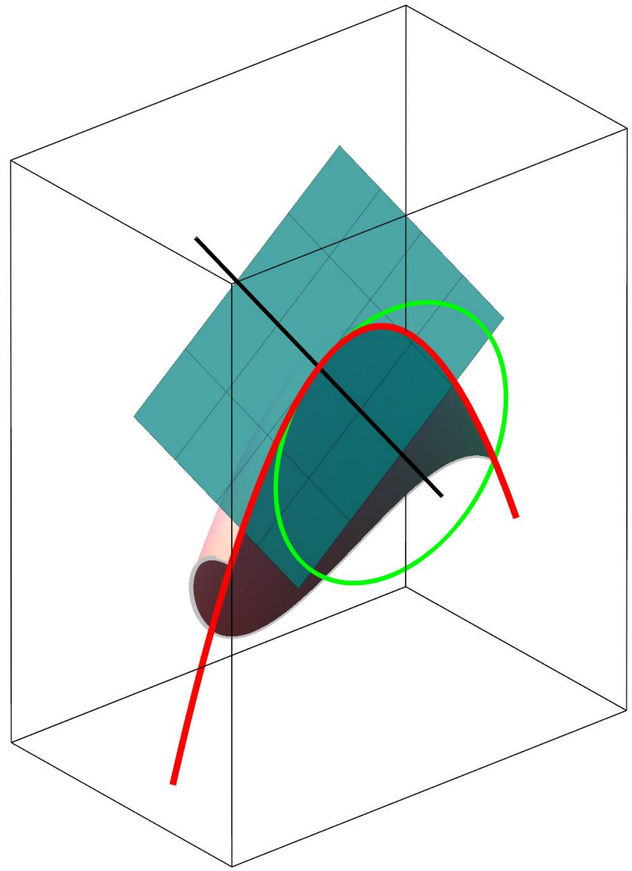

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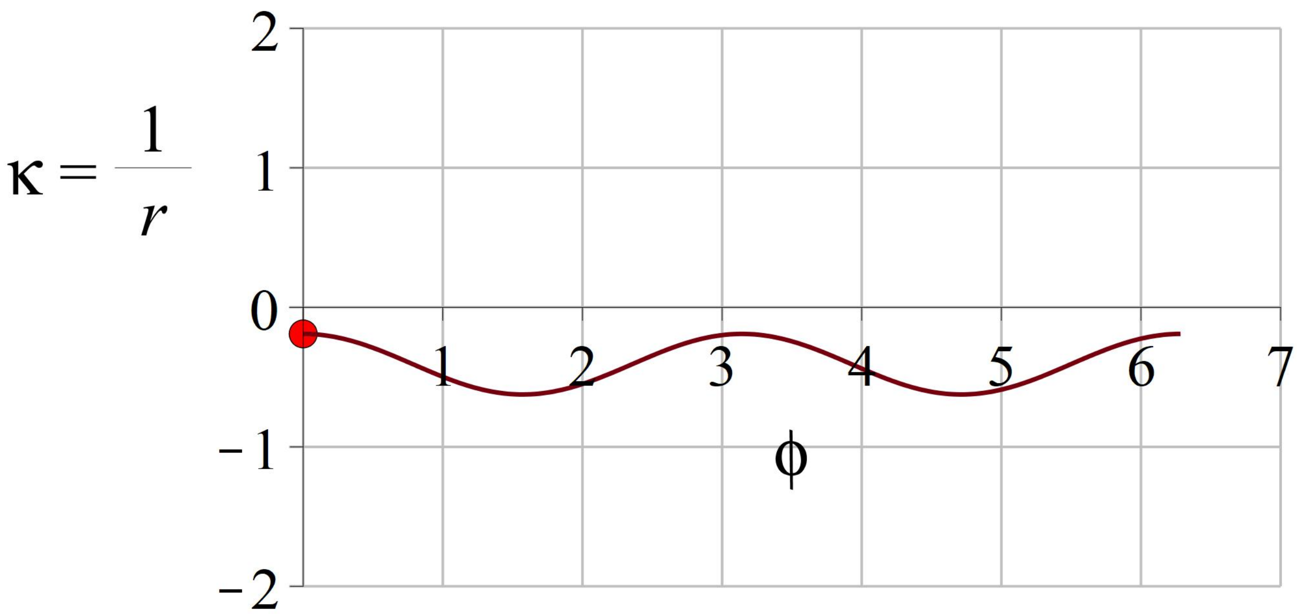


Curvature

Theorem 6 (Euler, Gauss, Riemann, et al.: Theorema Egregium)**.** *The so-called Gauss curvature* K *for a surface in 3D now has (at least) these quite different expressions at each point:*

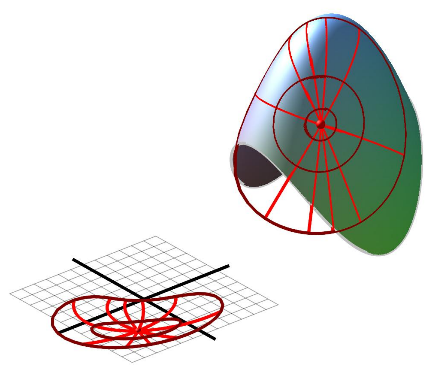
$$K = \max_{\theta}(\kappa(\theta)) \cdot \min_{\theta}(\kappa(\theta))$$

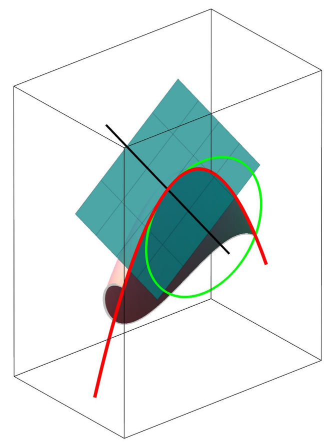
$$= \lim_{\rho \to 0} \left(\frac{3}{\pi}\right) \cdot \left(\frac{2\pi\rho - \mathcal{L}(\partial D(\rho))}{\rho^3}\right)$$

$$= \frac{\mathcal{R}(e_1, e_2, e_2, e_1)}{\mathrm{Area}^2(e_1, e_2)} \quad .$$

Curvature

The *geodesic spray circle* $\partial D(\rho)$ has length $\mathcal{L}(\partial D(\rho))$ that gives the curvature at the center point for $\rho \to 0$. The *Euler normal cutting procedure* at the point gives the normal curvatures $\kappa(\theta)$ and thence the alternative (extrinsic) max $\cdot$ min construction of the curvature:





Thank you for your attention!