INTRODUCTION TO DIFFERENTIAL GEOMETRY EXERCISES

1. Geodesics

Use symmetry and the fact that there is a unique geodesic (shortest curve) between any pair of (sufficiently close) points to prove (without calculations) that every great circle on a standard sphere in 3D is a geodesic and that there are no other geodesics on the sphere.

2. Distances

On a disk D with radius R in $\mathbb{R}^2$ with the standard Euclidean metric we consider the circular boundary ∂D parametrized by

$$\gamma(t) = (R \cdot \cos(t), R \cdot \sin(t)) \quad , \quad t \in \,]-\pi, \pi] \quad . \tag{2.1}$$

What is the boundary-to-boundary distance function

$$d : \quad \partial D \times \partial D \mapsto \mathbb{R}_+ \cup \{0\} \quad ? \tag{2.2}$$

3. Metric matrices

A parametrized surface in Euclidean $\mathbb{R}^3$ is given as follows:

$$r(x^1, x^2) = (\cos(x^1),\, \sin(x^1),\, x^2) \ , \ (x^1, x^2) \in \mathcal{U} = \,]-\pi, \pi[\, \times \mathbb{R} \quad .$$

Determine its induced metric matrix function $g_{i\,j}(x^1, x^2)$ and the corresponding Christoffel symbols $\Gamma^k_{i\,j}(x^1, x^2)$.

4. Conductivities

Show that the current vector $I = \mathcal{W}(\mathrm{grad}_g f)$ for a potential function f always has a non-zero component in the direction of $\mathrm{grad}_g f$ when $\mathrm{grad}_g(f)$ is non-zero and the conductivity tensor $\mathcal{W}$ is self-adjoint and positive definite with respect to g.