Exercises 2 (M.V.)

① Suppose ω is any Lipschitz subdomain of $B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$. Let R be a fixed number > 2. Let w be the solution to

$$\begin{cases} -\Delta w + w = 0 & \text{in } B_R \setminus \omega \\ w = 1 & \text{on } \omega \\ w = 0 & \text{on } \partial B_R \end{cases}$$

Prove that

$$c_R \|w\|_{H^1}^2 \leq \mathrm{cap}_D(\omega) \leq C_R \|w\|_{H^1}^2$$

with, constants c_R & C_R independent of ω

Hint: you may use that if $\varphi \in H^{1/2}(\partial B_R)$ then there exists $V_\varphi \in H^1(B_R \setminus B_1)$ so that $V_\varphi = 0$ on ∂B_1, $V_\varphi = \varphi$ on ∂B_R and $\|V_\varphi\|_{H^1} \leq C_R \|\varphi\|_{H^{1/2}}$

(2) Same setting as in problem (1).
Let w_0 denote the solution to

$$\begin{cases} -\Delta w_0 = 0 & \text{in } B_R \setminus \omega \\ w_0 = 1 & \text{on } \omega \\ w_0 = 0 & \text{on } \partial B_R \end{cases}$$

Prove that

$$c_R \|w_0\|_{H^1} \leq \|w\|_{H^1} \leq C_R \|w_0\|_{H^1}$$

with constants c_R & C_R independent of ω.

(3) Set $\omega = B_\varepsilon$, $\varepsilon \ll 1$, use the results from (1) & (2) to find the asymptotics of $\mathrm{cap}_D(B_\varepsilon)$ as $\varepsilon \searrow 0$. Let D_ε be the "disk"

$$D_\varepsilon = \{ x : x_d = 0 \quad |x'| \leq \varepsilon \}$$

(with $x' = (x_1, x_2 \dots x_{d-1})$). What can you say about $cap_D(P_E)$?