Excercises (M.V.)

(1) What happens to the representation formula for $u_\varepsilon - u_0$ if you replace the Neumann function $N(x,y)$ with another fundamental solution, f. ex.

$$\Phi(x,y) = \begin{cases} +\frac{1}{2\pi} \log |x-y| \\ -\frac{1}{4\pi} |x-y|^{-1} \end{cases}$$

(assuming γ_0 is constant)

(2) Show that the Neumann function $N(x,y)$ (normalized by $\int_{\partial\Omega} N(x,y)\,d\sigma_x = 0$) is symmetric in its arguments, i.e.,

$$N(x,y) = N(y,x) \qquad x,y \in \Omega$$

(3) Given $\varphi \in L^2_0(\partial\Omega) = L^2(\partial\Omega) \cap \{\int_{\partial\Omega} \varphi = 0\}$ let u_0 be the solution to

$$\Delta u_0 = 0 \text{ in } \Omega \subseteq \mathbb{R}^d, \quad \frac{\partial u_0}{\partial n} = \varphi \text{ on } \partial\Omega$$

Let $z_1, z_2 \cdots z_p$ be p different points in Ω. Consider the map

$$G: L^2_0(\partial\Omega) \ni \varphi \longrightarrow (\nabla u_0(z_1) \cdots \nabla u_0(z_p)) \in \mathbb{R}^{d \times p}$$

Show that G is a surjective linear map!