MINICOURSE ON GEOMETRIC INVERSE PROBLEMS

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Abstract. These are lecture notes for a minicourse on geometric inverse problems, to be given at the virtual DTU inverse problems winter school in January 2021.

Preface

Many fundamental inverse problems are formulated in Euclidean space. Such problems include

- determining a function in $\mathbb{R}^2$ from its integrals over straight lines (Radon transform inverse problem);
- determining the sound speed in a domain in $\mathbb{R}^n$ from boundary measurements of solutions of the wave equation (Gel'fand problem);
- determining the electrical conductivity in a domain in $\mathbb{R}^n$ from voltage and current measurements on its boundary (Calderón problem).

In this minicourse we will study inverse problems in *geometric*, or *non-Euclidean*, settings. For Radon transform problems this will mean that straight lines are replaced by more general curves. For Gel'fand or Calderón type problems this will mean that domains in $\mathbb{R}^n$ are replaced by more general geometric spaces.

A particularly clean setting, which is still relevant for several applications, is the one where domains in $\mathbb{R}^n$ are replaced by Riemannian manifolds and straight lines are replaced by geodesic curves of a smooth Riemannian metric. We will focus on this setting and formulate our questions on compact Riemannian manifolds (M, g) with smooth boundary. This corresponds to working with compactly supported functions in the Radon transform problem.

Notation. In these notes M will always be a compact, oriented, smooth $(= C^\infty)$ manifold with smooth boundary, and g will be a smooth Riemannian metric on M. We assume that $n = \dim(M) \geq 2$. We write $\langle \,\cdot\,, \,\cdot\, \rangle_g$ and $|\cdot|_g$ for the g-inner product and norm on tangent vectors. In local coordinates we write $g = (g_{jk})_{j,k=1}^n$, and (g^{jk}) is the inverse matrix of (g_{jk}). Thus if $x = (x_1, \ldots, x_n)$ are local coordinates and if $\partial_j = \frac{\partial}{\partial x_j}$ are the corresponding coordinate vector fields, then $g_{jk} = \langle \partial_j, \partial_k \rangle_g$ and

$$\langle X^j \partial_j, Y^k \partial_k \rangle_g = g_{jk} X^j Y_k, \qquad |X^j \partial_j|_g = (g_{jk} X^j X^k)^{1/2}.$$

Here and below we use the Einstein summation convention that a repeated upper and lower index is summed from 1 to n (i.e. we omit the sum signs).

We denote by $\nabla_g = \mathrm{grad}_g$ and by div_g the Riemannian gradient and divergence on M. The Laplace-Beltrami operator is $\Delta_g = \mathrm{div}_g \nabla_g$. In local coordinates one has the formulas

$$\begin{aligned} \nabla_g u &= g^{jk} \partial_j u \partial_k, \\ \mathrm{div}_g(X^j \partial_j) &= \det(g)^{-1/2} \partial_j (\det(g)^{1/2} X^j), \\ \Delta_g u &= \det(g)^{-1/2} \partial_j (\det(g)^{1/2} g^{jk} \partial_k u). \end{aligned}$$

We denote the volume form on (M, g) by dV_g, and the induced volume form on ∂M by dS_g. If $u, v \in C^\infty(M)$, one has the integration by parts (or Green) formula

$$\int_{\partial M} (\partial_\nu u) v \, dS_g = \int_M ((\Delta_g u) v + \langle \nabla_g u, \nabla_g v \rangle_g) \, dV_g$$

where ν is the outer unit normal vector to ∂M, and $\partial_\nu u = \langle \nabla_g u, \nu \rangle_g |_{\partial M}$ is the normal derivative on ∂M.

We note that we may drop the subindices g for brevity. All geodesics are assumed to have unit speed, i.e. to satisfy $|\dot\gamma(t)|_g = 1$.

1. Geodesic X-ray transform

In this section we discuss the geodesic X-ray transform, which generalizes the classical X-ray (or Radon) transform in Euclidean space. We will prove that the geodesic X-ray transform is injective on compact *simple* manifolds.

1.1. The Radon transform in $\mathbb{R}^2$.

To set the stage, we review a few facts about the classical Radon transform. See [He99, Na01] for further information.

The *X-ray transform* If of a function f in $\mathbb{R}^n$ encodes the integrals of f over all straight lines, whereas the *Radon transform* Rf encodes the integrals of f over $(n-1)$-dimensional planes. We will focus on the case $n = 2$, where the two transforms coincide. This transform appears naturally in medical imaging in X-ray computed tomography (CT) and positron emission tomography (PET).

There are many ways to parametrize the set of lines in $\mathbb{R}^2$. We will parametrize lines by their direction vector ω and signed distance s from the origin.

Definition. If $f \in C_c^\infty(\mathbb{R}^2)$, the *Radon transform* of f is the function

$$Rf(s,\omega) := \int_{-\infty}^{\infty} f(s\omega^\perp + t\omega)\,dt, \quad s \in \mathbb{R},\ \omega \in S^1.$$

Here $\omega^\perp$ is the vector in S^1 obtained by rotating ω counterclockwise by $90°$.

There is a well-known relation between Rf and the Fourier transform $\hat{f}$. We denote by $(Rf)\tilde{\ }(\,\cdot\,,\omega)$ the Fourier transform of Rf with respect to s.

Theorem 1.1 (Fourier slice theorem)**.**

$$(Rf)\tilde{\ }(\sigma,\omega) = \hat{f}(\sigma\omega^\perp).$$

Proof. Parametrizing $\mathbb{R}^2$ by $y = s\omega^\perp + t\omega$, we have

$$\begin{aligned}(Rf)\tilde{\ }(\sigma,\omega) &= \int_{-\infty}^{\infty} e^{-i\sigma s}\left[\int_{-\infty}^{\infty} f(s\omega^\perp + t\omega)\,dt\right]ds = \int_{\mathbb{R}^2} e^{-i\sigma y\cdot\omega^\perp} f(y)\,dy \\ &= \hat{f}(\sigma\omega^\perp).\end{aligned}$$

□

This result already proves that the Radon transform Rf uniquely determines f:

Theorem 1.2 (Uniqueness)**.** *If $f \in C_c^\infty(\mathbb{R}^2)$ is such that $Rf \equiv 0$, then $f \equiv 0$.*

Proof. If $Rf \equiv 0$, then $\hat{f} \equiv 0$ by Theorem 1.1 and consequently $f \equiv 0$ by the Fourier inversion theorem. □

The interplay between the Radon and Fourier transforms can further be used to study reconstruction algorithms and stability and range properties for the Radon transform inverse problem. The use of the Fourier transform is possible because the Euclidean space $\mathbb{R}^2$ is highly symmetric, and can be nicely tiled with straight lines. In more general geometric spaces, symmetries and Fourier methods may not be available so that one needs to employ different methods.

1.2. **The geodesic X-ray transform.** We will now introduce the geodesic X-ray transform following [PSU21, Chapters 3 and 4], see also [Sh94]. This transform appears in seismic and ultrasound imaging, e.g. as the linearization of the boundary rigidity/inverse kinematic problem. We will see in the later sections that it also arises in the study of inverse problems for partial differential equations.

Let (M,g) be a compact manifold with smooth boundary, assumed to be embedded in a compact manifold (N,g) without boundary. We parametrize geodesics by points in the *unit sphere bundle*, defined by

$$SM := \{(x,v)\,;\, x \in M,\ v \in T_xM,\ |v|_g = 1\}.$$

We also consider the unit spheres

$$S_xM := \{v \in T_xM \,;\, |v|_g = 1\}, \qquad x \in M.$$

If $(x,v) \in SN$ we denote by $\gamma_{x,v}(t)$ the geodesic in N which starts at the point x in direction v, that is,

$$D_{\dot\gamma}\dot\gamma = 0, \quad \gamma_{x,v}(0) = x, \quad \dot\gamma_{x,v}(0) = v.$$

Recall that the geodesic equation $D_{\dot\gamma}\dot\gamma = 0$ reads in local coordinates as

$$\ddot\gamma^l(t) + \Gamma^l_{jk}(\gamma(t))\dot\gamma^j(t)\dot\gamma^k(t) = 0$$

where $\Gamma^l_{jk} = \frac{1}{2}g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk})$ are the Christoffel symbols of the metric $g = (g_{jk})_{j,k=1}^n$, and (g^{jk}) is the inverse matrix of (g_{jk}).

We also denote by φ_t the *geodesic flow* on SN,

$$\varphi_t : SN \to SN, \quad \varphi_t(x,v) = (\gamma_{x,v}(t), \dot\gamma_{x,v}(t)).$$

If $(x,v) \in SM$ let $\tau(x,v) \in [0,\infty]$ be the first time when $\gamma_{x,v}(t)$ exits M,

$$\tau(x,v) := \sup\{t \geq 0 \,;\, \gamma_{x,v}([0,t]) \subset M\}.$$

We assume that (M,g) is *nontrapping*, meaning that $\tau(x,v)$ is finite for any $(x,v) \in SM$. (If $\tau(x,v) = \infty$, we say that the geodesic $\gamma_{x,v}$ is *trapped*.)

Definition. The *geodesic X-ray transform* of a function $f \in C^\infty(M)$ is defined by

$$If(x,v) := \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,dt, \quad (x,v) \in \partial(SM).$$

Thus, If encodes the integrals of f over all maximal geodesics in M starting from ∂M, such geodesics being parametrized by points of $\partial(SM) = \{(x,v) \in SM \,;\, x \in \partial M\}$.

So far we have not imposed any restrictions on the behavior of geodesics in (M,g) other than the nontrapping condition. However, invertibility of the geodesic X-ray transform is only known under certain geometric restrictions. One class of manifolds where such results have been proved is the following.

Definition. A compact Riemannian manifold (M,g) with smooth boundary is called *simple* if

(a) its boundary ∂M is strictly convex,
(b) it is nontrapping, and
(c) no geodesic has conjugate points.

We explain briefly the notions appearing in the definition:

1. (Strict convexity) We say that ∂M is *strictly convex* if the second fundamental form of ∂M in M is positive definite. This implies that any geodesic in N that is tangent to ∂M stays outside M for small positive and negative times. Thus any maximal geodesic going from ∂M into M stays inside M except for its endpoints, which corresponds to the usual notion of strict convexity in Euclidean space.

   We will only use the following consequence of (a): if ∂M is strictly convex, then the exit time function τ is C^∞ in SM^{int} and hence all functions in the analysis below are C^∞, see [PSU21, Section 3.2]. In fact assumption (a) could be removed with extra arguments [GMT17].

2. (Nontrapping) The *nontrapping* condition means that any geodesic in M should reach the boundary ∂M in finite time. An example of a trapped geodesic is the equator in a large spherical cap $\{x \in S^2\,;\, x_3 \geq -\varepsilon\}$.

3. (Conjugate points) If $\gamma : [a, b] \to M$ is a geodesic segment and if there is a family of geodesics $(\gamma_s)_{s\in(-\varepsilon,\varepsilon)}$ such that $\gamma_0 = \gamma$ and $\gamma_s(a) = \gamma(a)$, $\gamma_s(b) = \gamma(b)$ for $s \in (-\varepsilon, \varepsilon)$, then the points $\gamma(a)$ and $\gamma(b)$ are said to be *conjugate* along γ. This is a sufficient and almost necessary condition for conjugate points; for precise definitions see [PSU21, Section 3.7]. As an example, the north and south poles on the sphere are conjugate along any geodesic (=great circle) connecting them.

   Part (c) of the definition of a simple manifold states that there is no pair of conjugate points along any geodesic segment in M. Informally this means that there is no family of geodesics that starts at one point and converges to another point after some time. When $\dim(M) = 2$, a sufficient condition for no conjugate points is that the Gaussian curvature satisfies $K(x) \leq 0$ for all $x \in M$ (in higher dimensions it is enough that all sectional curvatures are nonpositive).

The class of simple manifolds turns out frequently in geometric inverse problems. There are several equivalent definitions [PSU21, Section 3.8] and we will use one of them later in these notes. We mention that any simple manifold is diffeomorphic to a ball, so one can think of M as being just the closed unit ball in $\mathbb{R}^n$ with some nontrivial Riemannian metric g.

Example 1.3 (Simple manifolds)**.** Strictly convex bounded smooth domains in $\mathbb{R}^n$, or in nonpositively curved Riemannian manifolds, are simple. An example with positive curvature is given by the spherical cap $M = \{x \in S^2\,;\, x_3 \geq \varepsilon\}$, where S^2 is the unit sphere in $\mathbb{R}^3$ and $\varepsilon > 0$. Note that such a spherical cap does not contain trapped geodesics or conjugate points. Small metric perturbations of simple manifolds are also simple.

The main result of this section, proved first in [Mu77] in two dimensions, states that the geodesic X-ray transform is injective on simple manifolds.

Theorem 1.4 (Injectivity)**.** *Let (M, g) be a simple manifold. If $f \in C^\infty(M)$ satisfies $If = 0$, then $f = 0$.*

We note that on general manifolds injectivity may fail:

Example 1.5 (Counterexamples)**.** There are two basic examples of manifolds where the geodesic X-ray transform is not injective. The first is a large spherical cap $M = \{x \in S^2\,;\, x_3 \geq -\varepsilon\}$. Any odd function f supported in a small neighborhood of e_1 and $-e_1$ integrates to zero over all great circles, hence $If = 0$ but f is nontrivial. Another example is a catenoid type surface with a flat cylinder glued in the middle. Note that both examples contain trapped geodesics. The latter example has no conjugate points.

Theorem 1.4 is still in a sense the best available result on the geodesic X-ray transform on two-dimensional manifolds. When $\dim(M) \geq 3$ further results are available, based on the microlocal method introduced in [UV16]. These results are valid on strictly convex nontrapping manifolds that admit a strictly convex function, i.e. a function $\varphi \in C^\infty(M)$ such that $\mathrm{Hess}_g(\varphi) > 0$, or more generally are foliated by strictly convex hypersurfaces. Such manifolds may have conjugate points. We also mention that the nontrapping condition can be weakened slightly [Gu17].

The following questions remain open (see the survey [IM19] for further references):

Question 1.1. *Is the geodesic X-ray transform injective on compact strictly convex nontrapping manifolds?*

Question 1.2. *Does every simple manifold admit a strictly convex function?*

Question 1.3. *Are there other examples of manifolds where the geodesic X-ray transform is not injective?*

In the rest of this section we will sketch a proof of Theorem 1.4 following the argument in [PSU13] under two simplifying assumptions:

- $\dim(M) = 2$ (to simplify the analysis on SM);
- $f \in C^\infty_c(M^{\mathrm{int}})$ (to remove regularity issues near ∂M).

The proof contains two parts:

1. Reduction from the integral equation $If = 0$ into a partial differential equation $VXu = 0$ on SM.
2. Uniqueness result for the equation $VXu = 0$ in SM based on energy methods.

1.3. **Reduction to PDE.** Assume that $f \in C_c^\infty(M^{\mathrm{int}})$ satisfies $If = 0$. We begin by introducing the primitive

$$u(x,v) = u^f(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt, \qquad (x,v) \in SM.$$

Here we think of f as a function on SM by taking $f(x,v) = f(x)$. Note that $u|_{\partial(SM)} = If = 0$. Since τ is smooth in SM^{int} and f vanishes near ∂M, we in fact have $u \in C_c^\infty(SM^{\mathrm{int}})$.

Next we introduce the *geodesic vector field* $X : C^\infty(SN) \to C^\infty(SN)$, which differentiates a function on SN along geodesic flow:

$$Xw(x,v) = \frac{d}{ds} w(\varphi_s(x,v))\Big|_{s=0}.$$

We note that the function $u = u^f$ above satisfies

$$\begin{aligned} Xu(x,v) &= \frac{d}{ds} u(\varphi_s(x,v))\Big|_{s=0} = \frac{d}{ds} \int_0^{\tau(\varphi_s(x,v))} f(\varphi_t(\varphi_s(x,v)))\,dt\Big|_{s=0} \\ &= \frac{d}{ds} \int_0^{\tau(x,v)-s} f(\varphi_{t+s}(x,v))\,dt\Big|_{s=0} \\ &= \frac{d}{ds} \int_s^{\tau(x,v)} f(\varphi_r(x,v))\,dr\Big|_{s=0} \\ &= -f(x). \end{aligned}$$

In particular we have

$$Xu = -f(x) \text{ on } SM, \qquad u|_{\partial SM} = If = 0. \tag{1.1}$$

The problem (1.1) can be considered as an *inverse source problem* for a transport equation: the source $f(x)$ in the equation produces a measurement $u|_{\partial(SM)} = If = 0$. We wish to prove uniqueness in the sense that if the measurement $u|_{\partial(SM)}$ is zero, then the source must be zero.

Note that the equation is on $SM = \{(x,v) \in TM\,;\, |v| = 1\}$, but the source $f(x)$ only depends on x and not on v. We can further get rid of the source by differentiating the equation $Xu(x,v) = -f(x)$ with respect to v. To do this in a coordinate-invariant way, we introduce the following notions:

Definition. Let (M,g) be an oriented two-dimensional manifold. Given $v \in S_xM$, we define $v^\perp$ (*rotation by* 90° *counterclockwise*) to be the unique vector in S_xM so that $(v, v^\perp)$ is a positively oriented orthonormal basis of T_xM. Morever, given $\theta \in (-\pi, \pi]$, we define the *rotation*

$$R_\theta v = (\cos\theta)v + (\sin\theta)v^\perp.$$

Finally, we define the *vertical vector field* $V : C^\infty(SM) \to C^\infty(SM)$ by

$$Vw(x,v) = \frac{d}{d\theta} w(R_\theta(x,v))\Big|_{\theta=0}, \qquad (x,v) \in SM.$$

Example 1.6 (X and V in the Euclidean disk)**.** Let $M = \overline{\mathbb{D}} \subset \mathbb{R}^2$ and let g be the Euclidean metric. Then

$$SM = \{(x, v_\theta)\,;\, x \in M,\ \theta \in (-\pi, \pi]\}$$

where $v_\theta = (\cos\theta, \sin\theta)$. We identify (x, v_θ) with (x, θ). Then

$$Xw(x,\theta) = \frac{d}{dt} w(x + tv_\theta, \theta)\Big|_{t=0} = v_\theta \cdot \nabla_x w(x,\theta)$$

and

$$Vw(x,\theta) = \frac{d}{d\theta} w(x,\theta).$$

If $f(x)$ is independent of v, clearly $Vf = 0$. Thus if $f \in C_c^\infty(M^{\mathrm{int}})$ satisfies $If = 0$, then by (1.1) the primitive $u = u^f \in C_c^\infty(SM^{\mathrm{int}})$ satisfies

$$VXu = 0 \text{ in } SM.$$

This reduces the geodesic X-ray transform problem to showing that the only solution of the equation $VXu = 0$ on SM which vanishes near ∂M is the zero solution.

1.4. **Uniqueness via energy methods.** The required uniqueness result will be a consequence of the following energy estimate.

Proposition 1.7 (Energy estimate)**.** *If (M, g) is a two-dimensional simple manifold, then*

$$\|Xu\|_{L^2(SM)} \leq \|VXu\|_{L^2(SM)}$$

for any $u \in C_c^\infty(SM^{\mathrm{int}})$.

The L^2 norm above is interpreted as follows. Recall that on any Riemannian manifold (M, g) there is a volume form dV_g. Moreover, if $x \in M$ the metric g induces an inner product (i.e. metric) $g(x)$ on T_xM, and hence a metric and volume form dS_x on the unit sphere S_xM. We then have the $L^2(SM)$ inner product

$$(u, w) = \int_{SM} u\bar{w}\, d\Sigma := \int_M \int_{S_xM} u(x,v)\overline{w(x,v)}\, dS_x(v)\, dV_g(x)$$

and the corresponding norm

$$\|u\| = \|u\|_{L^2(SM)} = \left(\int_{SM} |u|^2\, d\Sigma \right)^{1/2}.$$

The proof of the main theorem, when $\dim(M) = 2$ and $f \in C_c^\infty(M^{\mathrm{int}})$, follows easily from Proposition 1.7.

Proof of Theorem 1.4. Let $f \in C_c^\infty(M^{\mathrm{int}})$ satisfy $If = 0$. We have seen that the primitive $u = u^f$ is in $C_c^\infty(SM^{\mathrm{int}})$ and satisfies $VXu = 0$ in SM. Proposition 1.7 gives that $Xu = 0$ in SM. By (1.1) we get $f = -Xu = 0$. □

It remains to prove Proposition 1.7. Write

$$P := VX.$$

The equation $Pu = 0$ in SM is a second order PDE on the three-dimensional manifold SM. It does not belong to any of the standard classes (elliptic, parabolic, hyperbolic etc). Nevertheless we can prove an energy estimate for it by using a *positive commutator argument*.

We first need to compute the formal adjoint of P in the $L^2(SM)$ inner product. We start with the adjoints of X and V.

Lemma 1.8 (Adjoints of X and V)**.** *The vector fields X and V are formally skew-adjoint operators in the sense that*

$$(Xu, w) = -(u, Xw), \quad (Vu, w) = -(u, Vw)$$

for $u, w \in C_c^\infty(SM^{\mathrm{int}})$.

Assuming this, the formal adjoint of P is $P^* = (VX)^* = XV$. Thus we may decompose P in terms of its self-adjoint and skew-adjoint parts:

$$P = A + iB, \qquad A = \frac{P + P^*}{2}, \qquad B = \frac{P - P^*}{2i}. \tag{1.2}$$

(Compare with the decomposition $z = a + ib$ of a complex number into its real and imaginary parts.) Since $A^* = A$ and $B^* = B$, we can now study the norm $\|VXu\| = \|Pu\|$ for $u \in C_c^\infty(SM^{\mathrm{int}})$ as follows:

$$\begin{aligned}\|Pu\|^2 &= (Pu, Pu) = ((A + iB)u, (A + iB)u) \\ &= \|Au\|^2 + \|Bu\|^2 + i(Bu, Au) - i(Au, Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u) \qquad (1.3)\end{aligned}$$

where $[A, B] := AB - BA$ is the *commutator* of A and B.

In Proposition 1.7 we need to prove that $\|Pu\| \geq \|Xu\|$. We can obtain a lower bound for $\|Pu\|$ from (1.3) if the commutator term $(i[A, B]u, u)$ is positive (or if it can be absorbed in the positive terms $\|Au\|^2$ and $\|Bu\|^2$). The commutator has the form

$$\begin{aligned}2i[A, B] &= \frac{1}{2}[P + P^*, P - P^*] = [P^*, P] = P^*P - PP^* \\ &= XVVX - VXXV.\end{aligned}$$

To study this we need to commute X and V. Define the vector field

$$X_\perp := [X, V].$$

Lemma 1.9 (Commutator formulas)**.** *If (M,g) is two-dimensional, one has*

$$\begin{aligned}[X,V] &= X_\perp,\\ [V,X_\perp] &= X,\\ [X,X_\perp] &= -KV\end{aligned}$$

where K is the Gaussian curvature of (M,g).

Example 1.10 (Euclidean case)**.** Let $M = \overline{\mathbb{D}} \subset \mathbb{R}^2$ and let g be the Euclidean metric. As in Example 1.6 we may identify (x, v_θ) with (x,θ). Then $X_\perp$ has the form

$$\begin{aligned}X_\perp w &= XVw - VXw = v_\theta \cdot \nabla_x(\partial_\theta w) - \partial_\theta(v_\theta \cdot \nabla_x w)\\ &= -(\partial_\theta v_\theta)\cdot \nabla_x w = -v_\theta^\perp \cdot \nabla_x w.\end{aligned}$$

The formulas in Lemma 1.9 can be checked by direct computations, e.g.

$$\begin{aligned}[X,X_\perp]w &= XX_\perp w - X_\perp X w = v_\theta \cdot \nabla_x(-v_\theta^\perp \cdot \nabla_x w) + v_\theta^\perp \cdot \nabla_x(v_\theta \cdot \nabla_x w)\\ &= 0.\end{aligned}$$

This is consistent since $K = 0$ for the Euclidean metric. In general computing $[X, X_\perp]$ requires commuting two covariant derivatives, and hence one expects the curvature to appear.

We will indicate how to prove Lemmas 1.8 and 1.9 in the end of this section. Using Lemma 1.9, we can easily compute the commutator $i[A,B]$:

$$\begin{aligned}2i[A,B] &= XVVX - VXXV\\ &= VXVX + X_\perp VX - VXVX - VXX_\perp\\ &= VX_\perp X - XX - VXX_\perp\\ &= VKV - XX.\end{aligned}$$

Thus by Lemma 1.8

$$(2i[A,B]u,u) = \|Xu\|^2 - (KVu,Vu). \tag{1.4}$$

We observe:

- If g is the Euclidean metric, then $K \equiv 0$ and $(i[A,B]u,u) = \|Xu\|^2 \geq 0$.
- More generally if (M,g) has nonpositive curvature, i.e. $K \leq 0$, then $(i[A,B]u,u) \geq \|Xu\|^2 \geq 0$.

Going back to (1.3) and using that $\|Au\|^2 + \|Bu\|^2 \geq 0$, we see that if (M,g) is a two-dimensional simple manifold which additionally has nonpositive curvature, then

$$\|VXu\|^2 \geq \|Xu\|^2, \qquad u \in C_c^\infty(SM^{\mathrm{int}}).$$

This proves Proposition 1.7 in the (already nontrivial and interesting) case where $K \leq 0$.

To prove Proposition 1.7 in general we need to exploit the $\|Au\|^2$ and $\|Bu\|^2$ terms more carefully. Using (1.2) it is easy to check that

$$\begin{aligned}\|Au\|^2 + \|Bu\|^2 &= \frac{1}{4}\|(P+P^*)u\|^2 + \frac{1}{4}\|(P-P^*)u\|^2 \\ &= \frac{1}{2}\|Pu\|^2 + \frac{1}{2}\|P^*u\|^2.\end{aligned}$$

Inserting this back in (1.3) gives

$$\|Pu\|^2 = \|P^*u\|^2 + 2(i[A,B]u,u).$$

Since $P = VX$ and $P^* = XV$, using (1.4) yields the identity

$$\|VXu\|^2 = \|XVu\|^2 - (KVu, Vu) + \|Xu\|^2.$$

This is an important energy identity in the study of X-ray transforms, known as the *Pestov identity*. The proof of Proposition 1.7 is completed by the following lemma, which explicitly uses the no conjugate points assumption.

Lemma 1.11. *If (M,g) is a two-dimensional simple manifold, then*

$$\|XVu\|^2 - (KVu, Vu) \geq 0, \qquad u \in C_c^\infty(SM^{\mathrm{int}}).$$

Proof. If $\gamma : [0,\tau] \to M$ is a geodesic segment, we recall the *index form* (see [PSU21, Section 3.7])

$$I_\gamma(Y,Y) = \int_0^\tau (|D_tY(t)|_g^2 - K(\gamma(t))|Y(t)|_g^2)\,dt$$

defined for vector fields Y along γ that are normal to $\dot\gamma$. This is the bilinear form associated with the Jacobi equation $-D_t^2 J(t) - K(\gamma(t))J(t) = 0$. The basic property is that γ has no conjugate points iff $I_\gamma(Y,Y) > 0$ for all normal vector fields $Y \not\equiv 0$ along γ that vanish at the endpoints.

We will also need the *Santaló formula* (see [PSU21, Section 3.5]), which is a change of variables formula on SM and states that

$$\int_{SM} w\,d\Sigma = \int_{\partial_+ SM} \left[\int_0^{\tau(x,v)} w(\varphi_t(x,v))\,dt\right] \mu\, d(\partial SM)$$

where $\partial_+ SM = \{(x,v) \in \partial(SM)\,;\, \langle v,\nu\rangle_g \leq 0\}$ and $\mu = -\langle v,\nu\rangle_g$, with ν being the outward unit normal to ∂M. Applying this to $w = |XVu|^2 - K|Vu|^2$, and using for any $(x,v) \in \partial_+ SM$ the normal vector field

$$Y_{x,v}(t) := Vu(\varphi_t(x,v))\dot\gamma(t)^\perp$$

along $\gamma_{x,v}$, implies that

$$\begin{aligned}
&\|XVu\|^2 - (KVu, Vu) \\
&= \int_{\partial_+ SM} \left[\int_0^{\tau(x,v)} (|XVu(\varphi_t(x,v))|^2 - K(\gamma_{x,v}(t))|Vu(\varphi_t(x,v))|^2)\,dt \right] \mu\, d(\partial SM) \\
&= \int_{\partial_+ SM} \left[\int_0^{\tau(x,v)} (|D_t Y_{x,v}(t)|^2 - K(\gamma_{x,v}(t))|Y_{x,v}(t)|^2)\,dt \right] \mu\, d(\partial SM) \\
&= \int_{\partial_+ SM} I_{\gamma_{x,v}}(Y_{x,v}, Y_{x,v}) \mu\, d(\partial SM).
\end{aligned}$$

The last quantity is ≥ 0, since the index form is nonnegative by the no conjugate points condition. □

Remark 1.12. If (M, g) is simple and $n = \dim(M) \geq 3$, the same scheme as above can be used to prove that the geodesic X-ray transform is injective. However, the vector fields V and $X_\perp$ need to be replaced by suitable vertical and horizontal gradient operators $\overset{\mathtt{v}}{\nabla}$ and $\overset{\mathtt{h}}{\nabla}$, and the Pestov identity takes the form

$$\|\overset{\mathtt{v}}{\nabla} Xu\|^2 = \|X \overset{\mathtt{v}}{\nabla} u\|^2 - (R \overset{\mathtt{v}}{\nabla} u, \overset{\mathtt{v}}{\nabla} u) + (n-1)\|Xu\|^2$$

where $RZ(x, v) := R_x(Z, v)v$ is the Riemann curvature tensor. See [PSU15] for details.

Finally we discuss the proof of Lemmas 1.8 and 1.9. One way to prove them is via local coordinate computations. There is a particularly useful coordinate system for this, known as *isothermal coordinates*. The existence of global isothermal coordinates is part of the uniformization theorem for Riemann surfaces. It boils down to the following generalization of the Riemann mapping theorem from simply connected planar domains to simply connected Riemann surfaces. Here we use the basic fact that any simple manifold is simply connected, which follows by Morse theory [PSU21, Proposition 3.7.19].

Theorem 1.13 (Global isothermal coordinates)**.** *Let (M, g) be a compact oriented simply connected two-dimensional manifold with smooth boundary. There are global coordinates $x = (x_1, x_2)$ on M so that in these coordinates the metric has the form*

$$g_{jk}(x) = e^{2\lambda(x)} \delta_{jk}$$

for some real $\lambda \in C^\infty(M)$.

The isothermal coordinates induce global coordinates (x_1, x_2, θ) on SM where $\theta \in (-\pi, \pi]$ is the angle between v and $\partial/\partial x_1$, i.e.

$$v = e^{-\lambda(x)}(\cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2}).$$

Exercise 1.1. Let (M, g) be a compact oriented simply connected two-dimensional manifold with smooth boundary. Use the (x_1, x_2) and (x_1, x_2, θ) coordinates above to do the following (see [PSU21, Section 3.5] for hints if needed):

(a) Compute the Christoffel symbols $\Gamma^l_{jk}(x)$.

(b) Show that X, $X_\perp$ and V are given by

$$X = e^{-\lambda}\left(\cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + \left(-\frac{\partial\lambda}{\partial x_1}\sin\theta + \frac{\partial\lambda}{\partial x_2}\cos\theta\right)\frac{\partial}{\partial\theta}\right),$$

$$X_\perp = -e^{-\lambda}\left(-\sin\theta \frac{\partial}{\partial x_1} + \cos\theta \frac{\partial}{\partial x_2} - \left(\frac{\partial\lambda}{\partial x_1}\cos\theta + \frac{\partial\lambda}{\partial x_2}\sin\theta\right)\frac{\partial}{\partial\theta}\right),$$

$$V = \frac{\partial}{\partial\theta}.$$

Hint. To compute X, you can use the equation $\tan\theta(t) = \frac{\dot{x}_2(t)}{\dot{x}_1(t)}$ where $(x_1(t), x_2(t), \theta(t))$ is a geodesic in the (x_1, x_2, θ) coordinates.

(c) Prove Lemma 1.8. You can use (b) and the fact that

$$\int_{SM} w\, d\Sigma = \int_M \int_{-\pi}^{\pi} w(x,\theta) e^{2\lambda(x)}\, d\theta\, dx.$$

(d) Prove Lemma 1.9. You can use (b) and the fact that if $g_{jk}(x) = e^{2\lambda(x)}\delta_{jk}$, then the Gaussian curvature has the form

$$K = -\Delta_g \lambda = -e^{-2\lambda}(\partial_1^2\lambda + \partial_2^2\lambda).$$

2. Gel'fand problem

Seismic imaging gives rise to various inverse problems related to determining interior properties, e.g. oil deposits or deep structure, of the Earth. Often this is done by using acoustic or elastic waves. We will consider the following problem, which has many names and equivalent forms. It is also known as the *inverse boundary spectral problem* (see the monograph [KKL01]):

> **Gel'fand problem:** Is it possible to determine the interior structure of Earth by controlling acoustic waves and measuring vibrations at the surface?

In seismic imaging one often tries to recover an unknown sound speed. However, in this presentation we consider the simpler case where the sound speed is known and one attempts to recover an unknown potential q. We assume that the Earth is modelled by a compact Riemannian n-manifold (M, g) with smooth boundary (in practice M is a closed ball in $\mathbb{R}^3$), and the metric g models the sound speed. In fact, if $c(x)$ is a scalar sound speed in a domain in $\mathbb{R}^n$, the corresponding metric is

$$g_{jk}(x) = c(x)^{-2}\delta_{jk}.$$

A general metric g corresponds to an *anisotropic* (non-scalar) sound speed. Thus Riemannian geometry already appears when considering sound speeds in Euclidean domains.

Consider the free wave operator

$$\Box := \partial_t^2 - \Delta$$

in $M \times (0, T)$, where Δ is the Laplace-Beltrami operator in (M, g):

$$\Delta u = \mathrm{div}(\nabla u) = \det(g)^{-1/2}\partial_j(\det(g)^{1/2} g^{jk}\partial_k u).$$

Here the operators $\nabla = \nabla_g$, $\mathrm{div} = \mathrm{div}_g$, and $\Delta = \Delta_g$ only act in the x variable. Let also $q \in C_c^\infty(M^{\mathrm{int}})$ be a time-independent potential.

We assume that the medium is at rest at time $t = 0$ and that we take measurements until time $T > 0$. If we prescribe the amplitude of the wave to be $f(x, t)$ on $\partial M \times (0, T)$, this leads to a solution u of the wave equation

$$\left\{ \begin{array}{rll} (\Box + q)u &= 0 & \text{in } M \times (0, T), \\ u &= f & \text{on } \partial M \times (0, T), \\ u = \partial_t u &= 0 & \text{on } \{t = 0\}. \end{array} \right. \tag{2.1}$$

Given any $f \in C_c^\infty(\partial M \times (0, T))$, there is a unique solution $u \in C^\infty(M \times (0, T))$ (see [Ev10, Theorem 7 in §7.2.3] for the Euclidean case; the proof in the Riemannian case is the same). We assume that we can measure the normal derivative $\partial_\nu u|_{\partial M \times (0,T)}$, where $\partial_\nu u(x, t) = \langle \nabla u(x, t), \nu(x) \rangle$ and ν is the outer unit normal to ∂M. We do such measurements for many different functions f. The ideal boundary measurements are encoded by the *hyperbolic Dirichlet-to-Neumann map* (DN map for short)

$$\Lambda_q : C_c^\infty(\partial M \times (0, T)) \to C^\infty(\partial M \times (0, T)), \quad \Lambda_q(f) = \partial_\nu u|_{\partial M \times (0,T)}.$$

The Gel'fand problem for this model amounts to recovering q from the knowledge of the map Λ_q. We will prove the following classical result. For simplicity we assume that the potentials are compactly supported in M^{int}.

Theorem 2.1 (Uniqueness)**.** *Assume that (M, g) is simple. Let $T > 0$ be sufficiently large and assume that $q_1, q_2 \in C_c^\infty(M^{\mathrm{int}})$. If*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then $q_1 = q_2$ in M.

Remark 2.2. It is natural that one needs T to be sufficiently large in Theorem 2.1. By finite propagation speed the map Λ_q is unaffected if one changes q outside the set $\{x \in M\,;\, \mathrm{dist}(x, \partial M) < T/2\}$.[1] For our proof it is enough that T is larger than the length of the longest maximal geodesic in M.

If in Theorem 2.1 one drops the assumption that (M, g) is simple, it is still possible to prove that

$$\int_0^\ell q_1(\gamma(t))\,dt = \int_0^\ell q_2(\gamma(t))\,dt \tag{2.2}$$

whenever $\gamma : [0, \ell] \to M$ is a non-trapped maximal geodesic in M with $\ell < T$. We will prove (2.2) in the case where (M, g) is simple. It then follows from the injectivity of the geodesic X-ray transform, i.e. Theorem 1.4, that $q_1 = q_2$.

Theorem 2.1 is in fact true for a general compact manifold (M, g) under the sharp condition $T > 2\sup_{x\in M} \mathrm{dist}(x, \partial M)$. This and many other results for *time-independent* coefficients follow from the *Boundary Control method* introduced in [Be87], see [KKL01, La18] for further developments. However, there are several open questions when $q = q(x, t)$ is *time-dependent*. This case arises in inverse problems for nonlinear equations or in general relativity. In that case (and if one considers the analogous problem on $\partial M \times \mathbb{R}$ instead of $\partial M \times (0, T)$, see Exercise 2.1), instead of (2.2), our method which is based on *geometric optics solutions* gives that

$$\int_0^\ell q_1(\gamma(t), t+\sigma)\,dt = \int_0^\ell q_2(\gamma(t), t+\sigma)\,dt \tag{2.3}$$

whenever γ is a maximal geodesic as above and $\sigma \in \mathbb{R}$ is a time-delay parameter. This means that the *light ray transforms* of q_1 and q_2 are the same. The curves $(\gamma(t), t+\sigma)$ where γ is a geodesic in M are called *light rays*; they are lightlike, or null, geodesics for the *Lorentzian metric* $-dt^2 + g(x)$. When (M, g) is simple the invertibility of the light ray transform follows from invertibility of the geodesic X-ray transform, see Exercise 2.1.

More generally, instead of the wave operator $\Box = \partial_t^2 - \Delta$ corresponding to the product Lorentzian metric $-dt^2 + g(x)$ in $M \times \mathbb{R}$, one could consider a more general Lorentzian metric $\bar{g}$ (i.e. a symmetric 2-tensor field on $M \times \mathbb{R}$ that has one negative and n positive eigenvalues at each point) and the

[1] If u and $\tilde{u}$ solve (2.1) for potentials q and $\tilde{q}$ with the same Dirichlet data f, and if $q = \tilde{q}$ in $U := \{x \in M\,;\, \mathrm{dist}(x, \partial M) < T/2\}$, then $w := u - \tilde{u}$ solves $(\Box + q)w = F$ where $F := -(q - \tilde{q})\tilde{u}$ vanishes in U and also in $\{(x,t)\,;\, \mathrm{dist}(x, \partial M) > t\}$ since $\tilde{u}$ vanishes there. Moreover, $w = \partial_t w = 0$ on $\{t = 0\}$ and $w|_{\partial M \times (0,T)} = 0$. By finite speed of propagation $\partial_\nu w|_{\partial M \times (0,T)} = 0$. This proves that $\Lambda_q = \Lambda_{\tilde{q}}$.

corresponding wave operator $\Box_{\bar{g}}$. Inverse problems for $\Box_{\bar{g}}$ constitute a wave equation analogue of the anisotropic Calderón problem (see Section 3).

The following questions remain open:

Question 2.1. *Can one recover a time-dependent potential $q \in C_c^\infty(M \times \mathbb{R})$ from the hyperbolic DN map on $\partial M \times \mathbb{R}$ for a general compact Riemannian manifold (M, g) with boundary?*

Question 2.2. *For which Lorentzian metrics $\bar{g}$ is the light ray transform invertible?*

Question 2.3. *For which Lorentzian metrics $\bar{g}$ does one have uniqueness in the Gel'fand problem?*

See [AFO20, FIKO19, FIO19, St17] for recent results on the above questions. We also mention that for *nonlinear* wave equations better results are available, see e.g. [La18].

We now start the proof of Theorem 2.1. Alternative presentations may be found in the lecture notes [Ok18, Sa20] (the latter only in the Euclidean case), and similar results in much more general settings appear in [SY18, OSSU20]. The proof proceeds in four steps.

1. Derivation of an integral identity showing that if $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 - q_2$ is L^2-orthogonal to certain products of solutions.
2. Construction of special solutions that concentrate near a light ray $(\gamma(t), t + \sigma)$ for some $\sigma > 0$.
3. Proof of (2.2) by inserting the special solutions in the integral identity and taking a limit.
4. Inversion of the geodesic X-ray transform to prove that $q_1 = q_2$.

The first step is an integral identity.

Lemma 2.3 (Integral identity)**.** *Assume that $q_1, q_2 \in C^\infty(M)$. For any $f_1, f_2 \in C_c^\infty(\partial M \times (0, T))$, one has*

$$((\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2)_{L^2(\partial M \times (0,T))} = \int_M \int_0^T (q_1 - q_2) u_1 \bar{u}_2 \, dt \, dV$$

where u_1 solves (2.1) with $q = q_1$ and $f = f_1$, and u_2 solves an analogous problem with vanishing Cauchy data on $\{t = T\}$:

$$\left\{ \begin{aligned} (\Box + q_2) u_2 &= 0 && \text{in } M \times (0, T), \\ u_2 &= f_2 && \text{on } \partial M \times (0, T), \\ u_2 = \partial_t u_2 &= 0 && \text{on } \{t = T\}. \end{aligned} \right. \tag{2.4}$$

Proof. We first compute the formal adjoint of the DN map: one has

$$(\Lambda_q f, h)_{L^2(\partial M \times (0,T))} = (f, \Lambda_q^T h)_{L^2(\partial M \times (0,T))}$$

where $\Lambda_q^T h = \partial_\nu v|_{\partial M\times(0,T)}$ with v solving $(\Box+q)v = 0$ so that $v|_{\partial M\times(0,T)} = h$ and $v = \partial_t v = 0$ on $\{t = T\}$. To prove this, we let u be the solution of (2.1) and integrate by parts:

$$\begin{aligned}
(\Lambda_q f, h)_{L^2(\partial M\times(0,T))} &= \int_{\partial M}\int_0^T (\partial_\nu u)\bar{v}\,dt\,dS \\
&= \int_M\int_0^T (\langle\nabla u, \nabla\bar{v}\rangle + (\Delta u)\bar{v})\,dt\,dV \\
&= \int_M\int_0^T (\langle\nabla u, \nabla\bar{v}\rangle + (\partial_t^2 u + qu)\bar{v})\,dt\,dV \\
&= \int_M\int_0^T (\langle\nabla u, \nabla\bar{v}\rangle - \partial_t u\partial_t\bar{v} + qu\bar{v})\,dt\,dV \\
&= \int_M\int_0^T (\langle\nabla u, \nabla\bar{v}\rangle + u(\overline{\partial_t^2 v + qv}))\,dt\,dV \\
&= \int_M\int_0^T (\langle\nabla u, \nabla\bar{v}\rangle + u\Delta\bar{v})\,dt\,dV \\
&= \int_{\partial M}\int_0^T u\partial_\nu\bar{v}\,dt\,dS \\
&= (f, \Lambda_q^T h)_{L^2(\partial M\times(0,T))}.
\end{aligned}$$

Now, if u_1 and u_2 are as stated, the computation above gives

$$(\Lambda_{q_1} f_1, f_2)_{L^2(\partial M\times(0,T))} = \int_M\int_0^T (\langle\nabla u_1, \nabla\bar{u}_2\rangle - \partial_t u_1\partial_t\bar{u}_2 + q_1 u_1\bar{u}_2)\,dt\,dV$$

and

$$\begin{aligned}
(\Lambda_{q_2} f_1, f_2)_{L^2(\partial M\times(0,T))} &= (f_1, \Lambda_{q_2}^T f_2)_{L^2(\partial M\times(0,T))} \\
&= \int_\Omega\int_0^T (\langle\nabla u_1, \nabla\bar{u}_2\rangle - \partial_t u_1\partial_t\bar{u}_2 + q_2 u_1\bar{u}_2)\,dt\,dV.
\end{aligned}$$

The result follows by subtracting these two identities. □

If $\Lambda_{q_1} = \Lambda_{q_2}$, it follows from Lemma 2.3 that

$$\int_M\int_0^T (q_1 - q_2)u_1\bar{u}_2\,dt\,dV = 0$$

for all solutions u_1 and u_2 of the given type.

We will now start the construction of special solutions concentrating near a light ray $(\gamma(t), t+\sigma)$ where $\sigma > 0$ is a small time delay parameter. We use the method of *geometrical optics*, also known as the *WKB method*, and first look for approximate solutions using the ansatz

$$v(x,t) = e^{i\lambda\varphi(x,t)}a(x,t)$$

where $\lambda > 0$ is a large parameter, φ is a real phase function, and a is an amplitude supported near the curve $t \mapsto (\gamma(t), t+\sigma)$. A direct computation, given below, shows that

$$(\Box + q)v = e^{i\lambda\varphi}\left[\lambda^2 \left[|\nabla_x \varphi|_g^2 - (\partial_t \varphi)^2\right] a + i\lambda L a + (\Box + q)a\right]$$

where L is a certain first order differential operator. Now v is a good approximate solution if the right hand side is very small when λ is large. In particular, we want the λ^2 term to vanish, which means that the phase function φ should solve the *eikonal equation*

$$|\nabla_x \varphi|_g^2 - (\partial_t \varphi)^2 = 0.$$

We will show that when (M, g) is simple, the function $\varphi(x, t) := t - r$ is a solution where (ω, r) are Riemannian polar coordinates as in Lemma 2.7. We also show that by solving transport equations involving L one can obtain an amplitude a supported near the curve $t \mapsto (\gamma(t), t + \sigma)$ satisfying

$$\|i\lambda L a + (\Box + q)a\|_{L^\infty} \to 0 \text{ as } \lambda \to \infty.$$

Thus v is an approximate solution in the sense that $(\Box + q)v = o(1)$ as $\lambda \to \infty$. These approximate solutions can then be converted into exact solutions by solving a Dirichlet problem for the wave equation.

After the outline above, we give the precise statement regarding concentrating solutions.

Proposition 2.4 (Concentrating solutions)**.** *Assume that* $q \in C_c^\infty(M^{\mathrm{int}})$, *and let* $\gamma : [0, \ell] \to M$ *be a maximal geodesic in* M *with* $\ell < T$. *Let also* $\sigma > 0$ *be a small enough time delay parameter. For any* $\lambda \geq 1$ *there is a solution* $u = u_\lambda$ *of* $(\Box + q)u = 0$ *in* $M \times (0, T)$ *with* $u = \partial_t u = 0$ *on* $\{t = 0\}$, *such that for any* $\psi \in C_c^\infty(M \times [0, T])$ *one has*

$$\lim_{\lambda\to\infty} \int_M \int_0^T \psi |u|^2 \, dt \, dV = \int_0^\ell \psi(\gamma(t), t + \sigma) \, dt. \tag{2.5}$$

Moreover, if $\tilde{q} \in C_c^\infty(M^{\mathrm{int}})$, *there is a solution* $\tilde{u} = \tilde{u}_\lambda$ *of* $(\Box + \tilde{q})\tilde{u} = 0$ *in* $M \times (0, T)$ *with* $\tilde{u} = \partial_t \tilde{u} = 0$ *on* $\{t = T\}$, *such that for any* $\psi \in C_c^\infty(M \times [0, T])$ *one has*

$$\lim_{\lambda\to\infty} \int_M \int_0^T \psi u \bar{\tilde{u}} \, dt \, dV = \int_0^\ell \psi(\gamma(t), t + \sigma) \, dt. \tag{2.6}$$

Remark 2.5. The fact that one can construct solutions to the wave equation that concentrate near light rays $t \mapsto (\gamma(t), t + \sigma)$ is a consequence of *propagation of singularities*. This general phenomenon states that singularities of solutions for operators with real valued principal symbol p propagate along *null bicharacteristic curves*, i.e. integral curves of the Hamilton vector field H_p, in phase space. The principal symbol of the wave operator $\Box$

is $p(x, t, \xi, \tau) = -\tau^2 + |\xi|_g^2$, and the light rays are projections to the (x, t) variables of null bicharacteristic curves for $\Box$.

At this point it is easy to prove the main result:

Proof of Theorem 2.1. Using the assumption $\Lambda_{q_1} = \Lambda_{q_2}$ and Lemma 2.3, we have

$$\int_M \int_0^T (q_1 - q_2) u_1 \bar{u}_2 \, dt \, dV = 0 \tag{2.7}$$

for any solutions u_j of $(\Box + q_j)u_j = 0$ in $M \times (0, T)$ so that $u_1 = \partial_t u_1 = 0$ on $\{t = 0\}$, and $u_2 = \partial_t u_2 = 0$ on $\{t = T\}$.

Let $\gamma : [0, \ell] \to M$ be a maximal unit speed geodesic segment in M with $\ell < T$, let $\sigma > 0$ be small, and let $u_1 = u_{1,\lambda}$ be the solution constructed in Proposition 2.4 for the potential q_1 with $u_1 = \partial_t u_1 = 0$ on $\{t = 0\}$. Moreover, let $u_2 = u_{2,\lambda}$ be the solution constructed in the end of Proposition 2.4 for the potential q_2 with $u_2 = \partial_t u_2 = 0$ on $\{t = T\}$. Taking the limit as $\lambda \to \infty$ in (2.7) and using (2.6) with $\psi(x, t) = (q_1 - q_2)(x)$, we obtain that

$$\int_0^\tau (q_1 - q_2)(\gamma(t)) \, dt = 0.$$

This is true for any maximal geodesic γ in M with length $\ell < T$. If we assume that T is larger than the length of the longest maximal geodesic in M, it follows that the geodesic X-ray transform of $q_1 - q_2$ vanishes. By Theorem 1.4 we obtain that $q_1 = q_2$. $\Box$

We will now begin the proof of Proposition 2.4. For the construction of the phase function we will use the following fact about simple manifolds. It essentially states that a manifold is simple iff it admits global polar coordinates centered at any point.

Lemma 2.6 (Exponential map on simple manifolds)**.** *Let (M, g) be compact with strictly convex smooth boundary. Then (M, g) is simple iff there is an open manifold (U, g) containing M as a compact subdomain such that for any $p \in M$, the exponential map $\exp_p$ is a diffeomorphism from its maximal domain D_p in T_pU onto U.*

The proof that any simple manifold satisfies the condition in Lemma 2.6 requires geometric arguments and may be found in [PSU21, Section 3.8]. For the purposes of this section, we can just take the condition in Lemma 2.6 to be the *definition* of a simple manifold. It follows that any $x \in U$ can be uniquely written as

$$x = \exp_p(r\omega)$$

for some $r \geq 0$ and $\omega \in S^{n-1}$, with $r\omega \in D_p$. Thus we may identify $x \in U$ with (ω, r). The coordinates (ω, r) are called *Riemannian polar coordinates*, or *polar normal coordinates*, in (U, g). We will need the following property.

Lemma 2.7 (Riemannian polar coordinates)**.** *In the (ω, r) coordinates the metric has the form*

$$g(\omega, r) = \begin{pmatrix} 1 & 0 \\ 0 & g_0(\omega, r) \end{pmatrix}.$$

Proof. It is enough to prove that $\langle \partial_r, \partial_r \rangle = 1$ and $\langle \partial_r, w \rangle = 0$, where $w = \dot{\eta}(0)$ for any curve $\eta(t) = (r, \omega(t))$. Since ∂_r is the tangent vector of a unit speed geodesic starting at p, one has $\langle \partial_r, \partial_r \rangle = 1$. If $\eta(t)$ is a curve as above, the fact that $\langle \partial_r, w \rangle = 0$ is precisely the content of the Gauss lemma in Riemannian geometry (see e.g. [PSU21, Section 3.7]). □

We can now prove the result on concentrating solutions. The proof is quite elementary although a bit long.

Proof of Proposition 2.4. Let $\gamma : [0, \ell] \to M$ be a maximal unit speed geodesic in M with $\ell < T$, and let initially $\sigma \in (0, T - \ell)$.

We first construct an approximate solution $v = v_\lambda$ for the operator $\Box + q$, having the form

$$v(x, t) = e^{i\lambda\varphi(x,t)} a(x, t)$$

where φ is a real phase function, and a is an amplitude supported near the curve $t \mapsto (\gamma(t), t + \sigma)$. Note that

$$\partial_t(e^{i\lambda\varphi} u) = e^{i\lambda\varphi}(\partial_t + i\lambda\partial_t\varphi)u,$$
$$\partial_t^2(e^{i\lambda\varphi} u) = e^{i\lambda\varphi}(\partial_t + i\lambda\partial_t\varphi)^2 u$$

and similarly for the x-derivatives

$$\nabla(e^{i\lambda\varphi} u) = e^{i\lambda\varphi}(\nabla + i\lambda\nabla\varphi)u,$$
$$\mathrm{div}\nabla(e^{i\lambda\varphi} u) = e^{i\lambda\varphi}(\mathrm{div} + i\lambda\langle\nabla\varphi, \cdot\,\rangle)(\nabla + i\lambda\nabla\varphi)u.$$

We thus compute

$$\begin{aligned}(\Box + q)(e^{i\lambda\varphi} a) &= e^{i\lambda\varphi}((\partial_t + i\lambda\partial_t\varphi)^2 - (\mathrm{div}_x + i\lambda\langle\nabla_x\varphi, \cdot\,\rangle)(\nabla_x + i\lambda\nabla_x\varphi) + q)a \\ &= e^{i\lambda\varphi}\big[\lambda^2 \left[|\nabla_x\varphi|_g^2 - (\partial_t\varphi)^2\right] a \\ &\quad + i\lambda\left[2\partial_t\varphi\partial_t a - 2\langle\nabla_x\varphi, \nabla_x a\rangle + (\Box\varphi)a\right] + (\Box + q)a\big]. \end{aligned} \tag{2.8}$$

We would like to have $(\Box + q)(e^{i\lambda\varphi} a) = O(\lambda^{-1})$, so that $v = e^{i\lambda\varphi} a$ would indeed be an approximate solution when λ is large. To this end, we first choose φ so that the λ^2 term in (2.8) vanishes. This will be true if φ solves the *eikonal equation*

$$|\nabla_x\varphi|_g^2 - (\partial_t\varphi)^2 = 0. \tag{2.9}$$

We make the simple choice

$$\varphi(x,t) := t - \psi(x) \tag{2.10}$$

where $\psi \in C^\infty(M)$ should solve the equation

$$|\nabla\psi|_g^2 = 1. \tag{2.11}$$

This is another eikonal equation, now only in the x variables. We now invoke the assumption that (M,g) is *simple* and give an explicit solution of (2.11).

Let (U,g) be an open manifold as in Lemma 2.6 that contains M as a compact subdomain. Let η be the maximal geodesic in U with $\eta|_{[0,\ell]} = \gamma$ and, possibly after decreasing $\sigma > 0$, $p := \eta(-\sigma) \in U \setminus M$. By Lemma 2.6, if D_p is the maximal domain of $\exp_p$ in T_pU, then

$$\exp_p : D_p \to U$$

is a diffeomorphism. Thus any point $x \in U$ can be written uniquely as

$$x = \exp_p(r\omega)$$

for some $r \geq 0$ and $\omega \in S^{n-1}$ with $r\omega \in D_p$. Identifying x with (ω, r) gives global coordinates in U. We claim that

$$\psi(\omega, r) := r$$

is a smooth solution of (2.11) near M. Note first that ψ is smooth in M, since the origin of polar coordinates is outside M. Now the fact that ψ solves (2.11) follows immediately from Lemma 2.7 since

$$\langle \nabla\psi, \nabla\psi\rangle = \langle \partial_r, \partial_r\rangle = 1.$$

With the choice $\varphi(x,t) = t - \psi(x)$, we have (2.9) and thus the equation (2.8) becomes

$$(\Box + q)(e^{i\lambda\varphi} a) = e^{i\lambda\varphi}\left[i\lambda(La) + (\Box + q)a\right] \tag{2.12}$$

where L is the operator defined by

$$La := 2\partial_t\varphi\partial_t a - 2\langle \nabla_x\varphi, \nabla_x a\rangle + (\Box\varphi)a.$$

Clearly $\partial_t\varphi = 1$, and since $\psi(\omega, r) = r$ we obtain from Lemma 2.7 that

$$\langle \nabla_x\varphi, \nabla_x a\rangle = g^{jk}\partial_{x_j}\varphi\partial_{x_k} a = \partial_r a.$$

Writing $b := \Box\varphi$, the operator L simplifies to

$$La = 2(\partial_t + \partial_r)a + ba. \tag{2.13}$$

We next look for the amplitude a in the form

$$a = a_0 + \lambda^{-1} a_{-1}.$$

Inserting this to (2.8) and equating like powers of λ, we get

$$(\Box + q)(e^{i\lambda\varphi}a) = e^{i\lambda\varphi}\left[i\lambda(La_0) + [iLa_{-1} + (\Box+q)a_0] + \lambda^{-1}(\Box+q)a_{-1}\right]. \tag{2.14}$$

We would like the last expression to be $O(\lambda^{-1})$. This will hold if a_0 and a_{-1} satisfy the *transport equations*

$$\begin{cases} La_0 &= 0, \\ La_{-1} &= i(\Box+q)a_0. \end{cases} \tag{2.15}$$

It is not hard to solve these transport equations. To do this, it is convenient to consider new coordinates (ω, z, w) near $M \times (0,T)$, where

$$z = \frac{t+r}{2}, \qquad w = \frac{t-r}{2}. \tag{2.16}$$

Then L in (2.13) simplifies to $2\partial_z + b$ in the sense that

$$LF(x,t) = (2\partial_z \breve{F} + \breve{b}\breve{F})(\omega, \frac{t+r}{2}, \frac{t-r}{2})$$

where $\breve{F}$ corresponds to F in the new coordinates:

$$\breve{F}(\omega, z, w) := F(\omega, z-w, z+w).$$

Finally, we can use an integrating factor to get rid of $\breve{b}$. One has

$$LF(x,t) = 2c^{-1}\partial_z(c\breve{F})(\omega, \frac{t+r}{2}, \frac{t-r}{2}) \tag{2.17}$$

provided that $2\partial_z c = \breve{b}c$, which holds e.g. with the choice

$$c(\omega, z, w) := e^{\frac{1}{2}\int_0^z \breve{b}(\omega,s,w)\,ds}.$$

We can now solve the transport equations (2.15). By (2.17) the first transport equation reduces to

$$\partial_z(c\breve{a}_0) = 0.$$

Recall that we want our amplitude a to be supported near the curve $t \mapsto (\eta(t), t+\sigma)$ in the (x,t) coordinates. Recall also that the center p of our polar coordinates was given by $p = \eta(-\sigma)$. Thus $\eta(t) = (\omega_0, t+\sigma)$ for some $\omega_0 \in S^{n-1}$ in the (ω, r) coordinates, and at time $\sigma + t$ the amplitude should be supported near $(\omega_0, \sigma + t)$. Because of these facts, it makes sense to choose

$$\breve{a}_0(\omega, z, w) := c(\omega, z, w)^{-1}\chi(\omega, w).$$

where $\chi \in C_c^\infty(S^{n-1} \times \mathbb{R})$ is supported near $(\omega_0, 0)$. We will later choose χ to depend on λ. Note also that $\gamma(t)$ exits M when $t = \ell$, which means that

$$\breve{a}_0|_{M \times [\sigma+\ell+\varepsilon, \sigma+\ell+2\varepsilon]} = 0$$

for some $\varepsilon > 0$ if σ is chosen so small that $\sigma + \ell < T$. We set $\breve{a}_0 = 0$ for $t \in [\sigma+\ell+\varepsilon, T]$.

Next we choose

$$\breve{a}_{-1}(\omega, z, w) := -\frac{1}{2ic} \int_0^z c((\Box + q)a_0)^{\breve{}}(\omega, s, w)\, ds.$$

The functions a_0 and a_{-1} satisfy (2.15), and they vanish unless w is small (i.e. r is close to $t - \sigma$). Then (2.14) becomes

$$(\Box + q)(e^{i\lambda\varphi} a) = F_\lambda$$

where

$$F_\lambda := \lambda^{-1} e^{i\lambda\varphi}(\Box + q)a_{-1}.$$

Using the Cauchy-Schwarz inequality, one can check that

$$\begin{aligned} \|F_\lambda\|_{L^\infty(M\times(0,T))} &\leq \lambda^{-1}\|(\Box + q)a_{-1}\|_{L^\infty(M\times(0,T))} \\ &\leq C\lambda^{-1}\|\chi\|_{W^{4,\infty}(S^{n-1}\times\mathbb{R})} \end{aligned} \tag{2.18}$$

uniformly over $\lambda \geq 1$. This concludes the construction of the approximate solution $v = e^{i\lambda\varphi}a$.

We next find an exact solution $u = u_\lambda$ of (2.1) having the form

$$u = v + R$$

where R is a correction term. Note that for t close to 0, $v(\,\cdot\,, t)$ is supported near $p \notin M$ and hence $v = \partial_t v = 0$ on $\{t = 0\}$. Note also that $(\Box+q)v = F_\lambda$. Thus u will solve (2.1) for $f = v|_{\partial M\times(0,T)}$ if R solves

$$\left\{ \begin{array}{rll} (\Box + q)R &= -F_\lambda & \text{in } M \times (0,T), \\ R &= 0 & \text{on } \partial M \times (0,T), \\ R = \partial_t R &= 0 & \text{on } \{t = 0\}. \end{array} \right. \tag{2.19}$$

By the wellposedness of this problem (see [Ev10, Theorem 5 in §7.2.3] for the Euclidean case, again the proof in the Riemannian case is the same), there is a unique solution R with

$$\|R\|_{L^\infty((0,T);H^1(M))} \leq C\|F_\lambda\|_{L^2((0,T);L^2(M))} \leq C\lambda^{-1}\|\chi\|_{W^{4,\infty}}. \tag{2.20}$$

We now fix the choice of χ so that (2.5) will hold. Recall that $\chi \in C_c^\infty(S^{n-1} \times \mathbb{R})$ is supported near $(\omega_0, 0)$. We may parametrize a neighborhood of ω_0 in S^{n-1} by points $y' \in \mathbb{R}^{n-1}$ so that ω_0 corresponds to 0, and thus we may think of χ as a function in $\mathbb{R}^n$ supported near 0. Let $\zeta \in C_c^\infty(\mathbb{R}^n)$ satisfy $\zeta = 1$ near 0 and $\|\zeta\|_{L^2(\mathbb{R}^n)} = 1$, and choose

$$\chi(y) := \varepsilon^{-n/2}\zeta(y/\varepsilon)$$

where

$$\varepsilon = \varepsilon(\lambda) = \lambda^{-\frac{1}{n+8}}.$$

With this choice

$$\|\chi\|_{L^2(\mathbb{R}^n)} = 1, \qquad \|\chi\|_{W^{4,\infty}(\mathbb{R}^n)} \lesssim \varepsilon^{-n/2-4} \lesssim \lambda^{1/2}.$$

It follows from (2.20) that

$$\|v\|_{L^2(M\times(0,T))} \lesssim 1, \qquad \|R\|_{L^2(M\times(0,T))} \lesssim \lambda^{-1/2}.$$

Since $u = v + R$, the integral in (2.5) has the form

$$\begin{aligned}\int_M \int_0^T \psi|u|^2 \, dV \, dt &= \int_M \int_0^T \psi|v|^2 \, dV \, dt + O(\lambda^{-1/2}) \\ &= \int_M \int_0^T \psi|a_0|^2 \, dV \, dt + O(\lambda^{-1/2}).\end{aligned}$$

Using that $\psi|a_0|^2$ is compactly supported in $M^{\mathrm{int}} \times (0,T)$, we may use the (y', r, t) coordinates (where $y' \in \mathbb{R}^{n-1}$ corresponds to $\omega \in S^{n-1}$) to see that

$$\begin{aligned}&\int_M \int_0^T \psi|u|^2 \, dV \, dt = \int_{\mathbb{R}^{n+1}} \psi(y', r, t)\varepsilon^{-n}\zeta(\frac{y'}{\varepsilon}, \frac{t-r}{2\varepsilon})^2 \, dy' \, dr \, dt + O(\lambda^{-1/2}) \\ &= \int_{\mathbb{R}^{n+1}} \psi(y', z-w, z+w)\varepsilon^{-n}\zeta(y'/\varepsilon, w/\varepsilon)^2 \, dy' \, dz \, dw + O(\lambda^{-1/2})\end{aligned}$$

by changing variables as in (2.16). Finally, changing y' to $\varepsilon y'$ and w to εw and letting $\lambda \to \infty$ (so $\varepsilon \to 0$) yields

$$\begin{aligned}\lim_{\lambda\to\infty} \int_M \int_0^T \psi|u|^2 \, dV \, dt &= \int_{\mathbb{R}^{n+1}} \psi(0', z, z)\zeta(y', w)^2 \, dy' \, dz \, dw \\ &= \int_{-\infty}^{\infty} \psi(0', z, z) \, dz\end{aligned}$$

by the normalization $\|\zeta\|_{L^2(\mathbb{R}^n)} = 1$ and the fact that $\psi \in C_c^\infty(M^{\mathrm{int}} \times [0,T])$. Undoing the changes of coordinates, we see that the curve $(0', z, z)$ in the (y', r, t) coordinates corresponds to $t \mapsto (\omega_0, t, t)$ in the (ω, r, t) coordinates. Thus

$$\int_{-\infty}^{\infty} \psi(0', z, z) \, dz = \int_0^\ell \psi(\gamma(t), t+\sigma) \, dt$$

which proves (2.5).

It remains to prove (2.6). Since $\gamma(t)$ exits M after time $\ell < T$, we have $v|_{M\times[\sigma+\ell+\varepsilon,\sigma+\ell+2\varepsilon]} = 0$ for some small $\varepsilon > 0$. Redefining v to be zero for $t \geq \sigma+\ell+2\varepsilon$, we see that (2.18) still holds. Then we choose R solving (2.19) but with $R = \partial_R R = 0$ on $\{t = T\}$ instead of $\{t = 0\}$. We can do such a construction for the potential $\tilde{q}$ instead of q. Since φ and a_0 are independent of the potential q, the same argument as above proves (2.6). □

Exercise 2.1 (Time-dependent case). Let $q \in C_c^\infty(M^{\mathrm{int}} \times \mathbb{R})$, and consider the Dirichlet problem

$$\left\{\begin{aligned} (\Box + q)u &= 0 && \text{in } M \times \mathbb{R}, \\ u &= f && \text{on } \partial M \times \mathbb{R}, \\ u &= 0 && \text{for } t \ll 0. \end{aligned}\right. \tag{2.21}$$

Here $t \ll 0$ means that $t \leq -T_0$ for some $T_0 \geq 0$. You may assume that this problem is well-posed and for any $f \in C_c^\infty(\partial M \times \mathbb{R})$ there is a unique solution $u \in C^\infty(M \times \mathbb{R})$. Consider the hyperbolic DN map

$$\Lambda_q : C_c^\infty(\partial M \times \mathbb{R}) \to C^\infty(\partial M \times \mathbb{R}), \quad f \mapsto \partial_\nu u|_{\partial M \times \mathbb{R}}.$$

(a) Formulate a counterpart of Lemma 2.3 in this case.
(b) Formulate a counterpart of Proposition 2.4. Which parts of the proof need to be modified?
(c) Use parts (a) and (b) to show that if $\Lambda_{q_1} = \Lambda_{q_2}$, then

$$\int_0^\ell q_1(\gamma(t), t+\sigma)\, dt = \int_0^\ell q_2(\gamma(t), t+\sigma)\, dt$$

for any maximal geodesic $\gamma : [0, \ell] \to M$ and any $\sigma \in \mathbb{R}$.
(d) Use the Fourier transform in σ and injectivity of the geodesic X-ray transform in (M, g) to invert the light ray transform in part (c) and to prove that $q_1 = q_2$. (*Hint.* Look at the derivatives of the Fourier transform at 0.)

3. Calderón problem

Electrical Impedance Tomography (EIT) is an imaging method with applications in seismic and medical imaging and nondestructive testing. The method is based on the following important inverse problem.

> **Calderón problem:** Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

In a standard formulation the medium is modelled by a bounded domain $\Omega \subset \mathbb{R}^n$ (in practice $n = 3$), and one considers boundary measurements for solutions of the conductivity equation

$$\mathrm{div}(\gamma \nabla u) = 0 \text{ in } \Omega$$

where $\gamma \in C^\infty(\overline{\Omega})$ is a positive function (electrical conductivity).

If the electrical properties of the medium depend on direction, which happens e.g. in muscle tissue, the medium is said to be *anisotropic* and $\gamma = (\gamma^{jk})$ is a positive definite matrix function. When $n \geq 3$ one can write $\gamma^{jk} = \det(g)^{1/2} g^{jk}$ for some Riemannian metric g, and the conductivity equation becomes

$$\mathrm{div}_g(\nabla_g u) = 0.$$

Thus Riemannian geometry appears already when considering anisotropic conductivities in Euclidean domains. More generally, if (M, g) is a compact manifold with smooth boundary, we can consider the equation

$$\mathrm{div}_g(\gamma \nabla_g u) = 0 \tag{3.1}$$

for a positive function $\gamma \in C^\infty(M)$. This equation contains both equations above as a special case.

As a final reduction, if we replace u by $\gamma^{-1/2}u$ in (3.1), we obtain the equivalent *Schrödinger equation*

$$(-\Delta_g + q)u = 0 \text{ in } M$$

where $q = \frac{\Delta_g(\gamma^{1/2})}{\gamma^{1/2}}$. It is this equation that we will study.

Let (M,g) be a compact manifold with smooth boundary, and let $q \in C^\infty(M)$ be a potential. Consider the Dirichlet problem

$$\left\{ \begin{aligned} (-\Delta_g + q)u &= 0 \qquad \text{in } M, \\ u &= f \qquad \text{on } \partial M. \end{aligned} \right. \tag{3.2}$$

We assume that 0 is not a Dirichlet eigenvalue. Then for any $f \in C^\infty(\partial M)$ there is a unique solution $u \in C^\infty(M)$. The boundary measurements are given by the (elliptic) DN map

$$\Lambda_q : C^\infty(\partial M) \to C^\infty(\partial M), \quad \Lambda_q f = \partial_\nu u|_{\partial M}.$$

The Calderón problem in this setting is to determine the potential q from the knowledge of the DN map Λ_q, when the metric g is known.

The Calderón problem is by now well understood in Euclidean domains [KV84, SU87, Na96, Bu08]. Moreover, if $\dim(M) = 2$ and M is simply connected then isothermal coordinates, see Theorem 1.13, can be used to reduce the Riemannian case to the Euclidean case. We will thus assume from now on that $\dim(M) \geq 3$. In this case the problem is open in general, but there are results in special product geometries.

Definition. We say that (M,g) is *transversally anisotropic* if

$$(M,g) \subset\subset (\mathbb{R} \times M_0, g), \qquad g = e \oplus g_0,$$

where $(\mathbb{R}, e)$ is the Euclidean line and (M_0, g_0) is a compact $(n-1)$-manifold with boundary called the *transversal manifold*.

The definition means that (M,g) is contained in a product manifold $\mathbb{R} \times M_0$ with coordinates (t,x) where $t \in \mathbb{R}$ and $x \in M_0$, and the metric looks like

$$g(t,x) = \left(\begin{array}{cc} 1 & 0 \\ 0 & g_0(x) \end{array} \right).$$

The Laplace-Beltrami operator has the form

$$-\Delta_g = -\partial_t^2 - \Delta_x$$

where Δ_x is the Laplace-Beltrami operator of (M_0, g_0). Note that this looks similar to the Gel'fand problem studied in Section 2, where we studied the wave operator $\partial_t^2 - \Delta_x$. Formally the *Wick rotation*, i.e. the map $t \mapsto it$, converts one equation to the other.

It turns out that, surprisingly, there are in fact analogies between the elliptic and hyperbolic inverse problems. One has the following counterpart of Theorem 2.1, first proved in [DKSU09].

Theorem 3.1 (Uniqueness)**.** *Let (M, g) be a compact transversally anisotropic manifold. Assume also that the transversal manifold (M_0, g_0) is simple. If $q_1, q_2 \in C^\infty(M)$ and if*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then $q_1 = q_2$ in M.

By conformal invariance Theorem 3.1 holds more generally for metrics of the form $g = c(e \oplus g_0)$ for $c \in C^\infty(M)$ positive. Moreover, the assumption that (M_0, g_0) is simple can be relaxed to the assumption that (M_0, g_0) has injective geodesic X-ray transform [DKLS16]. However, the following questions remain open:

Question 3.1. *Is Theorem 3.1 true for any transversal manifold (M_0, g_0)?*

Question 3.2. *Is Theorem 3.1 true for any compact manifold (M, g)?*

Similarly as for the wave equation, it turns out that one can get better results for *nonlinear* elliptic equations. Consider the model equation

$$\left\{ \begin{aligned} -\Delta_g u + q u^3 &= 0 \qquad \text{in } M, \\ u &= f \qquad \text{on } \partial M. \end{aligned} \right. \tag{3.3}$$

In fact the method applies to the nonlinearities qu^m for any integer $m \geq 3$. If $f \in C^\infty(\partial M)$ is small (say in the $C^{2,\alpha}(\partial M)$ norm), a Banach fixed point argument implies that (3.3) has a unique *small* solution $u \in C^\infty(M)$. One can define the *nonlinear DN map*

$$\Lambda_q^{\mathrm{NL}} : \{f \in C^\infty(\partial M) \,;\, \|f\|_{C^{2,\alpha}(\partial M)} < \delta\} \to C^\infty(\partial M), \quad \Lambda_q f = \partial_\nu u|_{\partial M}.$$

It was proved independently in [FO20, LLLS20] that Question 3.1, which is open for the linear Schrödinger equation, can be solved for the nonlinear equation (3.3).

Theorem 3.2 (Nonlinear case)**.** *Let (M, g) be a compact transversally anisotropic manifold, and let $q_1, q_2 \in C^\infty(M)$. If*

$$\Lambda_{q_1}^{\mathrm{NL}} = \Lambda_{q_2}^{\mathrm{NL}},$$

then $q_1 = q_2$ in M.

Let us now sketch the proof of Theorem 3.1. The general scheme will be exactly the same as in the proof of Theorem 2.1 in the wave equation case, but with a few important differences. The proof proceeds in four steps:

1. Derivation of an integral identity showing that if $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 - q_2$ is L^2-orthogonal to certain products of solutions.
2. Construction of special solutions that concentrate near *two-dimensional manifolds* $\mathbb{R} \times \gamma$ where γ is a maximal geodesic in M_0.
3. Inserting the special solutions in the integral identity and taking a limit, in order to recover integrals over geodesics.
4. Inversion of the geodesic X-ray transform to prove that $q_1 = q_2$.

3.1. **Integral identity.** The first step, the integral identity, is completely analogous to the wave equation case.

Lemma 3.3 (Integral identity)**.** *Let (M, g) be a compact manifold with boundary and let $q_1, q_2 \in C^\infty(M)$. If $f_1, f_2 \in C^\infty(\partial M)$, then*

$$((\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2)_{L^2(\partial M)} = \int_M (q_1 - q_2) u_1 \bar{u}_2 \, dV$$

where $u_j \in C^\infty(M)$ solves $(-\Delta + q_j)u_j = 0$ in M with $u_j|_{\partial M} = f_j$.

Proof. We first observe that the DN map is symmetric: if $q \in C^\infty(M)$ is real valued and if u_f solves $(-\Delta + q)u_f = 0$ in M with $u_f|_{\partial M} = f$, then an integration by parts shows that

$$\begin{aligned}(\Lambda_q f, g)_{L^2(\partial M)} &= \int_{\partial M} (\partial_\nu u_f)\overline{u}_g \, dS = \int_M (\langle \nabla u_f, \nabla \overline{u}_g \rangle + (\Delta u_f)\overline{u}_g) \, dV \\ &= \int_M (\langle \nabla u_f, \nabla \overline{u}_g \rangle + q u_f \overline{u}_g) \, dV \\ &= \int_{\partial M} u_f \overline{\partial_\nu u_g} \, dS = (f, \Lambda_q g)_{L^2(\partial M)}.\end{aligned}$$

Thus

$$\begin{aligned}(\Lambda_{q_1} f_1, f_2)_{L^2(\partial M)} &= \int_M (\langle \nabla u_1, \nabla \overline{u}_2 \rangle + q_1 u_1 \overline{u}_2) \, dV, \\ (\Lambda_{q_2} f_1, f_2)_{L^2(\partial M)} &= (f_1, \Lambda_{q_2} f_2)_{L^2(\partial M)} = \int_M (\langle \nabla u_1, \nabla \overline{u}_2 \rangle + q_2 u_1 \overline{u}_2) \, dV.\end{aligned}$$

The result follows by subtracting the above two identities. □

By Lemma 3.3, if $\Lambda_{q_1} = \Lambda_{q_2}$, then

$$\int_M (q_1 - q_2) u_1 \bar{u}_2 \, dV = 0 \tag{3.4}$$

for any solutions $u_j \in C^\infty(M)$ with $(-\Delta + q_j)u_j = 0$ in M.

3.2. **Special solutions and proof of Theorem 3.1.** We will next construct special solutions to the equation $(-\Delta + q)u = 0$. Just like for the wave equation, we start with the geometric optics ansatz

$$v(t, x) = e^{i\lambda\Phi(t,x)}a(t, x) \tag{3.5}$$

where $\lambda \in \mathbb{R}$ is a large parameter, Φ is a *complex valued* phase function, and a is an amplitude. The fact that the equation is elliptic requires us to use complex phase functions, and the corresponding solutions are called *complex geometrical optics solutions*.

The construction of special solutions is similar to the wave equation case, but has the following important differences which are consistent with the Wick rotation $t \mapsto it$:

- The eikonal equation is $\langle \nabla_x \Phi, \nabla_x \Phi\rangle_{g_0} + (\partial_t \Phi)^2 = 0$ instead of $|\nabla_x \varphi|_g^2 - (\partial_t \varphi)^2 = 0$. The phase function $\Phi(x, t) = it - r$ is complex valued, instead of being real valued as in $\varphi(x, t) = t - r$.
- The amplitude solves a complex transport equation $2(\partial_r + i\partial_t)a + ba = 0$, which has solutions concentrating near two-manifolds, instead of solving a real transport equation $2(\partial_r + \partial_t)a_0 + ba_0 = 0$ which has solutions concentrating near curves.
- The solutions concentrate near two-dimensional manifolds $\mathbb{R} \times \gamma$ where γ is a maximal geodesic in M_0, instead of concentrating near curves $t \mapsto (\gamma(t), t + \sigma)$.
- The approximate solutions $v = e^{i\lambda\Phi}a$ may grow exponentially in λ. Thus the exact solution $u = v + R$ cannot be constructed by solving a Dirichlet problem for R, but one must use a different solvability result (Carleman estimate).

We now discuss the argument in more detail. After applying the operator $-\Delta + q = -\partial_t^2 - \Delta_x + q(t, x)$ to the ansatz (3.5), we obtain a direct analogue of the wave equation computation (2.8):

$$\begin{aligned}(-\partial_t^2 - \Delta_x + q)(e^{i\lambda\Phi}a) = e^{i\lambda\Phi}\big[\lambda^2 \left[\langle \nabla_x \Phi, \nabla_x \Phi\rangle_{g_0} + (\partial_t \Phi)^2\right] a \\ - i\lambda \left[2\partial_t \Phi \partial_t a + 2\langle \nabla_x \Phi, \nabla_x a\rangle + (\Delta_{t,x}\Phi)a\right] + (-\Delta_{t,x} + q)a\big].\end{aligned} \tag{3.6}$$

Recall from (2.9) that in the wave equation case, the eikonal equation was

$$|\nabla_x \varphi|_g^2 - (\partial_t \varphi)^2 = 0$$

and we used the solution

$$\varphi = t - r$$

where (r, ω) were Riemannian polar coordinates in a neighborhood U of the simple manifold M_0, with center outside M_0. Recall also that we were interested in solutions that concentrate near the geodesic $\gamma : r \mapsto (r, \omega_0)$ in

M_0, where ω_0 is fixed. In the elliptic case, the eikonal equation appearing in the λ^2 term in (3.6) is

$$\langle \nabla_x \Phi, \nabla_x \Phi \rangle_{g_0} + (\partial_t \Phi)^2 = 0 \text{ in } M$$

and we obtain a solution by choosing

$$\Phi(t,x) := it - r.$$

This is consistent with the Wick rotation $t \mapsto it$.

Having solved the eikonal equation, (3.6) becomes

$$(-\Delta + q)(e^{i\lambda\Phi} a) = e^{i\lambda\Phi}(-i\lambda L a + (-\Delta + q)a)$$

where L is the complex first order operator

$$La := 2(\partial_r + i\partial_t)a + ba$$

where $b := \Delta_{t,x}\Phi$. Here $\partial_r + i\partial_t$ is a Cauchy-Riemann, or $\overline{\partial}$, operator. We wish to find an amplitude solving

$$La = 0 \text{ in } M.$$

Using coordinates (t, r, ω) where (r, ω) are polar coordinates as above, we choose the solution

$$a(t,r,\omega) = c(t,r,\omega)^{-1}\chi(\omega)$$

where c is an integrating factor solving $2(\partial_r + i\partial_t)c = bc$ (this amounts to solving a $\overline{\partial}$ equation in $\mathbb{R}^2$), and $\chi \in C_c^\infty(S^{n-2})$ is supported near ω_0.

We have produced a function $v = e^{i\lambda\Phi}a$ satisfying

$$(-\Delta + q)v = e^{i\lambda\Phi}(-\Delta + q)a$$

so that a is supported near the set two-dimensional manifold (t, r, ω_0), which corresponds to the set $\mathbb{R} \times \gamma$ where γ is a geodesic in M_0. As in Section 2 one could try to find an exact solution $u = v + R$ of $(-\Delta + q)u = 0$ in M by solving the Dirichlet problem

$$\left\{ \begin{array}{rll} (-\Delta + q)R &= -e^{i\lambda\Phi}(-\Delta + q)a & \text{in } M, \\ R &= 0 & \text{on } \partial M. \end{array} \right. \tag{3.7}$$

Now *if* Φ were real valued, the right hand side would be $O_{L^2(M)}(1)$ as $\lambda \to \infty$ and at least one would get a correction term $R = O_{L^2(M)}(1)$. This could be converted to an estimate $R = O_{L^2(M)}(\lambda^{-1})$ by working with an amplitude $a = a_0 + \lambda^{-1}a_{-1}$ as in Section 2.

However, the phase function is *not* real valued and in fact $e^{i\lambda\Phi} = e^{-\lambda t}e^{-i\lambda r}$. Thus the right hand side above is in general only $O(e^{C\lambda})$, which is not good since we wish to take the limit $\lambda \to \infty$. Instead of solving the Dirichlet problem, we need to use a different solvability result.

Proposition 3.4 (Carleman estimate)**.** *Let (M, g) be transversally anisotropic and let $q \in C^\infty(M)$. There are $C, \lambda_0 > 0$ so that whenever $|\lambda| \geq \lambda_0$ and $f \in L^2(M)$, there is a function $R \in H^1(M)$ satisfying*

$$(-\Delta + q)(e^{i\lambda\Phi} R) = e^{i\lambda\Phi} f \text{ in } M$$

such that

$$\|R\|_{L^2(M)} \leq \frac{C}{|\lambda|} \|f\|_{L^2(M)}.$$

Proof. See e.g. [DKSU09]. □

We can use Proposition 3.4 to convert the approximate solution $v = e^{i\lambda\Phi} a$ to an exact solution

$$u = e^{i\lambda\Phi}(a + R)$$

of $(-\Delta + q)u = 0$ in M, so that $\|R\|_{L^2(M)} \to 0$ as $|\lambda| \to \infty$. When $|\lambda|$ is large, the solution u is concentrated near the two-dimensional manifold

$$(\mathbb{R} \times \gamma) \cap M$$

but may grow exponentially in λ. However, the integral identity in Lemma 3.3 involves the product of two solutions, and we may take another solution of the type $e^{-i\lambda\Phi}(\tilde{a} + \tilde{R})$ so that the exponential growth will be cancelled in the product. By choosing such solutions and letting $\lambda \to \infty$ in (3.4), we obtain that the integral of $q_1 - q_2$ over the two-dimensional manifold $\mathbb{R} \times \gamma$ vanishes:

$$\int_{-\infty}^{\infty} \int_0^\ell (q_1 - q_2)(t, \gamma(r))\, dr\, dt = 0.$$

This is true for any maximal geodesic γ in (M_0, g_0), and hence using the injectivity of the geodesic X-ray transform on (M_0, g_0) would give that

$$\int_{-\infty}^{\infty} (q_1 - q_2)(t, x)\, dt = 0 \qquad \text{for all } x \in M_0.$$

This is not quite enough to conclude that $q_1 = q_2$. However, we can introduce an additional parameter $\sigma \in \mathbb{R}$, which is analogous to the time delay parameter in the wave equation case. This can be done by performing the above construction with *slightly complex frequency* $\lambda + i\sigma$. One obtains the following result:

Proposition 3.5 (Concentrating solutions)**.** *Let (M, g) be a transversally anisotropic manifold and let $q_1, q_2 \in C^\infty(M)$. Assume that the transversal manifold (M_0, g_0) is simple, and that $\gamma : [0, \ell] \to M_0$ is a maximal geodesic. There is $\lambda_0 > 0$ so that whenever $|\lambda| \geq \lambda_0$ and $\sigma \in \mathbb{R}$, there are solutions*

$u_1 = u_{1,\lambda}$ of $(-\Delta + q_1)u_1 = 0$ in M and $u_2 = u_{2,-\lambda}$ of $(-\Delta + q_2)u_2 = 0$ in M such that for any $\psi \in C_c^\infty(M^{\mathrm{int}})$ one has

$$\lim_{\lambda\to\infty} \int_M \psi u_1 \bar{u}_2 \, dV = \int_{-\infty}^{\infty} \int_0^{\ell} e^{-i\sigma(t+ir)} \psi(t, \gamma(r)) \, dr \, dt.$$

Theorem 3.1 now follows by inserting the solutions in Proposition 3.5 to the identity (3.4), taking the limit $\lambda \to \infty$, and using the Fourier transform in t and injectivity of the geodesic X-ray transform in (M_0, g_0) as in Exercise 2.1.

3.3. **Nonlinear case.** We will now consider the nonlinear equation

$$\left\{ \begin{array}{rl} -\Delta_g u + q u^3 = 0 & \text{in } M, \\ u = f & \text{on } \partial M \end{array} \right. \tag{3.8}$$

and the corresponding *nonlinear DN map*

$$\Lambda_q^{\mathrm{NL}} : \{ f \in C^\infty(\partial M) \,;\, \|f\|_{C^{2,\alpha}(\partial M)} < \delta \} \to C^\infty(\partial M), \quad \Lambda_q f = \partial_\nu u|_{\partial M}.$$

We will prove Theorem 3.2 which states that if $\Lambda_{q_1}^{\mathrm{NL}} = \Lambda_{q_2}^{\mathrm{NL}}$, then $q_1 = q_2$.

A standard method for dealing with inverse problems for nonlinear equations is *linearization*. Namely, if one knows the nonlinear DN map $\Lambda_q^{NL}(f)$ for small f, then one also knows its linearization or Frechet derivative

$$(D\Lambda_q^{NL})_0(h) = \partial_\varepsilon \Lambda_q^{NL}(\varepsilon h)|_{\varepsilon=0}, \qquad h \in C^\infty(M).$$

Let u_ε be the small solution of (3.8) with boundary value $f = \varepsilon h$, i.e.

$$\left\{ \begin{array}{rl} -\Delta_g u_\varepsilon + q u_\varepsilon^3 = 0 & \text{in } M, \\ u_\varepsilon = \varepsilon h & \text{on } \partial M. \end{array} \right. \tag{3.9}$$

Note that $u_0 = 0$, since $u = 0$ is the unique small solution with boundary value 0. Formally differentiating (3.9) in ε gives that

$$-\Delta_g(\partial_\varepsilon u_\varepsilon) + 3 q u_\varepsilon^2 \partial_\varepsilon u_\varepsilon = 0.$$

Setting $\varepsilon = 0$ and using that $u_0 = 0$, we see that

$$v_h := \partial_\varepsilon u_\varepsilon|_{\varepsilon=0}$$

solves the linear equation

$$\left\{ \begin{array}{rl} -\Delta_g v_h = 0 & \text{in } M, \\ v_h = h & \text{on } \partial M. \end{array} \right. \tag{3.10}$$

Thus the linearized solution v_h is just the harmonic function in (M, g) with boundary value h. This formal computation can be justified. Since

$$(D\Lambda_q^{NL})_0(h) = \partial_\varepsilon \Lambda_q^{NL}(\varepsilon h)|_{\varepsilon=0} = \partial_\varepsilon \partial_\nu u_\varepsilon|_{\varepsilon=0} = \partial_\nu v_h$$

this leads to the following:

Lemma 3.6 (Linearization of nonlinear DN map)**.**

$$(D\Lambda_q^{NL})_0(h) = \Lambda_g h$$

where Λ_g *is the DN map for the Laplace equation* (3.10)

This shows that from the knowledge of Λ_q^{NL}, we can recover its linearization $(D\Lambda_q^{NL})_0 = \Lambda_g$. However, this *first linearization* does not contain any information about the unknown potential q. It turns out that for the nonlinearity qu^3, the right thing to do is to look at the *third linearization*, i.e. the *third order Frechet derivative*, $(D^3\Lambda_q^{NL})_0$.

The third linearization can be computed by considering Dirichlet data of the form $f = \varepsilon_1 h_1 + \varepsilon_2 h_2 + \varepsilon_3 h_3$ where $h_j \in C^\infty(\partial M)$. Writing $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, let u_ε be the solution of

$$\left\{ \begin{array}{rll} -\Delta_g u_\varepsilon + q u_\varepsilon^3 &= 0 & \text{in } M, \\ u_\varepsilon &= \varepsilon_1 h_1 + \varepsilon_2 h_2 + \varepsilon_3 h_3 & \text{on } \partial M. \end{array} \right. \tag{3.11}$$

We formally apply the derivative $\partial_{\varepsilon_1\varepsilon_2\varepsilon_3}$ to this equation to obtain

$$\begin{aligned} 0 &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + q\partial_{\varepsilon_1\varepsilon_2\varepsilon_3}(u_\varepsilon^3) \\ &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + q\partial_{\varepsilon_1\varepsilon_2}(3u_\varepsilon^2 \partial_{\varepsilon_3} u_\varepsilon) \\ &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + q\partial_{\varepsilon_1}(6u_\varepsilon \partial_{\varepsilon_2} u_\varepsilon \partial_{\varepsilon_3} u_\varepsilon + 3u_\varepsilon^2 \partial_{\varepsilon_2\varepsilon_3} u_\varepsilon) \\ &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + 6q\partial_{\varepsilon_1} u_\varepsilon \partial_{\varepsilon_2} u_\varepsilon \partial_{\varepsilon_3} u_\varepsilon + \ldots \end{aligned}$$

where $\ldots$ consists of terms that contain a power of u_ε. Since $u_0 = 0$, when we set $\varepsilon = 0$ all the terms in $\ldots$ will vanish. Thus

$$w := \partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon|_{\varepsilon=0}$$

will solve the equation (recall that $v_{h_j} = \partial_{\varepsilon_j} u_\varepsilon|_{\varepsilon=0}$)

$$\left\{ \begin{array}{rll} -\Delta_g w &= -6q v_{h_1} v_{h_2} v_{h_3} & \text{in } M, \\ w &= 0 & \text{on } \partial M. \end{array} \right. \tag{3.12}$$

Now if the know the nonlinear DN map $\Lambda_q^{\mathrm{NL}}(\varepsilon_1 h_1 + \varepsilon_2 h_2 + \varepsilon_3 h_3) = \partial_\nu u_\varepsilon$, then we also know $\partial_\nu w = \partial_\nu \partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon|_\varepsilon = 0$. Thus for any $h_4 \in C^\infty(\partial M)$, we also know

$$\int_{\partial M} (\partial_\nu w) h_4 \, dS = \int_M ((\Delta_g w) v_{h_4} + \langle \nabla w, \nabla v_{h_4} \rangle_g) \, dV.$$

Integrating by parts in the last term, and using that $w|_{\partial M} = 0$ and $\Delta_g v_{h_4} = 0$, we obtain that

$$\int_{\partial M} (\partial_\nu w) h_4 \, dS = 6 \int_M q v_{h_1} v_{h_2} v_{h_3} v_{h_4} \, dV.$$

Since $\partial_\nu w$ is determined by Λ_q^{NL}, also the right hand side is determined by Λ_q^{NL}. (One can check that the left hand side is equal to $((D^3\Lambda_q^{NL})_0(h_1, h_2, h_3), h_4)_{L^2(\partial M)}$, where $(D^3\Lambda_q^{NL})_0$ is the third Frechet derivative of Λ_q^{NL} considered as a trilinear form.) This formal argument can be justified and it leads to the following identity:

Lemma 3.7 (Integral identity in nonlinear case)**.** *If* $\Lambda_{q_1}^{\mathrm{NL}} = \Lambda_{q_2}^{\mathrm{NL}}$*, then*

$$\int_M (q_1 - q_2) v_{h_1} v_{h_2} v_{h_3} v_{h_4} \, dV = 0$$

for all $h_1, h_2, h_3, h_4 \in C^\infty(\partial M)$*.*

This integral identity related to the nonlinear equation $-\Delta_g u + qu^3 = 0$ has two benefits over the identity for the linear equation $-\Delta_g u + qu = 0$:

- $q_1 - q_2$ is L^2-orthogonal to products of *four* solutions, instead of products of two solutions;
- the solutions v_{h_j} are solutions of the Laplace equation $\Delta_g v_{h_j} = 0$, which does not contain the potential q.

Let us finally sketch how one proves Theorem 3.2 based on the integral identity in Lemma 3.7 and the construction of special solutions in Proposition 3.5. The main point is that instead of considering a fixed geodesic γ in (M_0, g_0), one can consider *two intersecting geodesics.*

Suppose that γ_1 and γ_2 are two maximal geodesics in (M_0, g_0) that intersect only at one point $x_0 \in M_0$. We use Proposition 3.5 to find two harmonic functions v_λ and $v_{-\lambda}$ in M so that the product $v_\lambda \overline{v}_{-\lambda}$ is concentrated near $\mathbb{R} \times \gamma_1$. We similarly choose two harmonic functions w_λ and $w_{-\lambda}$ in M so that the product $w_\lambda \overline{w}_{-\lambda}$ is concentrated near $\mathbb{R} \times \gamma_2$. Then the product

$$v_\lambda \overline{v}_{-\lambda} w_\lambda \overline{w}_{-\lambda}$$

is concentrated near the *one-dimensional manifold* $\mathbb{R} \times \{x_0\}$, and one has

$$0 = \lim_{\lambda \to \infty} \int_M (q_1 - q_2) v_\lambda \overline{v}_{-\lambda} w_\lambda \overline{w}_{-\lambda} \, dV = \int_{-\infty}^{\infty} e^{-i\sigma t} (q_1 - q_2)(t, x_0) \, dt.$$

The point is that one has concentration at a *single point* x_0 in M_0, instead of concentration near a fixed geodesic in M_0. It follows that the Fourier transform of $(q_1 - q_2)(\,\cdot\,, x_0)$ vanishes identically for every $x_0 \in M_0$. It follows that $q_1 = q_2$.

In general, given $x_0 \in M_0$ it may not be possible to find two finite length geodesics that only intersect at x_0. The possibility of multiple intersection points can be handled by introducing another extra parameter in the solutions, see [LLLS20] for details. This proves Theorem 3.2 in general.

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